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I.—ON THE EVALUATION OF DEFINITE MULTIPLE INTEGRALS.

By R. L. ELLIS, M.A., Fellow of Trinity College.

THE following pages contain some general results obtained by means of Fourier's theorem. A few words will be sufficient to explain the manner in which it has been applied.

A definite multiple integral, where the limits are given by the inequality

$$f(xy.\ldots) \overline{>} h_1 \overline{<} h,$$

may be treated as if the limits of the different variables were independent of one another, provided the function under the signs of integration be considered discontinuous, and equal to zero whenever $f(xy.\ldots)$ transgresses the assigned limits h_1 and h. This idea has been made use of by M. Lejeune Dirichlet. It had, however, occurred to me before I was acquainted with his paper on multiple integrals, and the way in which I have applied it is, I believe, new.

Suppose the function to be integrated were of the form $\phi\,(xy.\ldots)\,\psi\{f(xy.\ldots)\}$: the limits being given by the inequality already mentioned. Then, by Fourier's theorem, writing $f.$ for $f(xy.\ldots)$,

$$\psi f. = \frac{1}{\pi}\int_0^\infty da \int_{h_1}^{h} \psi u.\ \cos a\,(f. - u)\,du,$$

for all values of $f.$ which lie between h_1 and h; moreover for the purposes of integration the formula may be applied, except in particular cases, so as to include these limiting values. (See Poisson, *Théorie de la Chaleur*); while for all

values of $f.$ which lie without these limits, the second side of the equation is equal to zero. Consequently the integral

$$\int dx \int dy \ldots \phi(xy\ldots)\,\psi\{f(xy\ldots)\}$$
$$= \frac{1}{\pi}\int_{h_1}^{h} \psi u du \int_0^\infty d\alpha \int dx \int dy \ldots \phi(xy\ldots) \cos \alpha (f. - u) \ldots\ldots (1);$$

the limits on the first side being given in the manner already mentioned. Those of the integrations with respect to x, y, &c. are arbitrary, provided they include all values of the variables which satisfy the given inequality.

Again, if in the given multiple integral the limits were determined by the single relation $f. \leqq h$, joined to the conditions that x, y, &c. were to have no values less than certain assigned limits; *e.g.* if we were to consider only positive values of the variables, the formula (1) would still apply with a slight modification. The inferior limit of integration with respect to u would be arbitrary, provided it included all the values which could be given to $f.$ by admissible values of the variables, while the inferior limits of integration for x, y, &c. would be determined by the particular conditions of the case.

Let us take, as an example of the method, the integral

$$\int dx \int dy \ldots x^{a-1} y^{b-1} z^{c-1} \ldots f(mx + ny + \ldots) \ldots\ldots (A),$$

m, n, &c. being all positive, and the limits being given by

$$mx + ny + \ldots \leqq h,$$

no negative values of the variables being admitted. In this case (1) becomes

$$\int dx \int dy \ldots x^{a-1} y^{b-1} \ldots f(mx + ny + \ldots)$$
$$= \frac{1}{\pi}\int_{h_1}^{h} f u du \int_0^\infty d\alpha \int_0 dx \int_0 dy \ldots x^{a-1} y^{b-1} \ldots \cos \alpha (mx + ny + \ldots - u) \ldots (2),$$

and the integrations with respect to x, y, &c. may be conveniently extended to infinity; (h_1 it should be observed is an arbitrary quantity < 0).

I first seek the value of

$$\int_0^\infty dx \int_0^\infty dy \ldots x^{a-1} y^{b-1} \ldots \cos \alpha (mx + ny + \ldots - u) = (B).$$

Integrating first for x, we get

$$(B) = \frac{\Gamma(a)}{m^a \alpha^a}\int_0^\infty dy \ldots y^{b-1} \ldots \cos\left\{a\frac{\pi}{2} + \alpha(ny + \ldots - u)\right\}.$$

This follows from the formulæ

$$\left.\begin{aligned}\int_0^\infty x^{a-1}\cos\alpha x dx &= \frac{\Gamma(a)}{\alpha^a}\cos a\frac{\pi}{2}\\ \int_0^\infty x^{a-1}\sin\alpha x dx &= \frac{\Gamma(a)}{\alpha^a}\sin a\frac{\pi}{2}\end{aligned}\right\}\dots\dots(\gamma).$$

Integrating in a similar manner for y, &c. successively, we get ultimately

$$(\mathrm{B}) = \frac{\Gamma(a)\,\Gamma(b)\dots}{m^a n^b\dots\alpha^{a+b+\dots}}\cos\left\{(a+b+\dots)\frac{\pi}{2}-\alpha u\right\}.$$

Now by (γ), we easily see that

$$\int_0^\infty v^{a+b+\dots-1}\cos\alpha(v-u)\,dv = \frac{\Gamma(a+b+\dots)}{\alpha^{a+b+\dots}}\cos\left\{(a+b+\dots)\frac{\pi}{2}-\alpha u\right\},$$

and therefore

$$(\mathrm{B}) = \frac{1}{m^a n^b\dots\dots}\cdot\frac{\Gamma(a)\,\Gamma(b)\dots\dots}{\Gamma(a+b+\dots\dots)}\int_0^\infty v^{a+b+\dots\dots-1}\cos\alpha(v-u)\,dv.$$

Hence $\displaystyle\int_0^\infty d\alpha\,(\mathrm{B}) = \frac{\pi}{m^a n^b\dots\dots}\cdot\frac{\Gamma(a)\,\Gamma(b)\dots\dots}{\Gamma(a+b+\dots\dots)}\,u^{a+b+\dots\dots-1}$,

for all positive values of u, and $= 0$ for all negative values, by Fourier's theorem. Consequently the second side of (2) becomes

$$\frac{1}{m^a n^b\dots\dots}\,\frac{\Gamma(a)\,\Gamma(b)\dots\dots}{\Gamma(a+b+\dots\dots)}\int_0^h fu u^{a+b+\dots\dots-1}du + \frac{1}{\pi}\int_{h_1}^0 fu.0.du.$$

Thus finally

$$\int dx\int dy\dots\dots x^{a-1}\,y^{b-1}\dots\dots f(mx+ny+\dots\dots)$$
$$= \frac{1}{m^a n^b\dots\dots}\,\frac{\Gamma(a)\,\Gamma(b)\dots\dots}{\Gamma(a+b+\dots\dots)}\int_0^h fu\,u^{a+b+\dots\dots-1}\,du\;\dots\dots(3);$$

which is equivalent to Liouville's extension of Dirichlet's theorem.

I proceed to evaluate the definite integral

$$\int_0 dx\int_0 dy\dots\dots e^{-(max+nby+\dots)}f(mx+ny+\dots\dots)\dots\dots(\mathrm{E}),$$

$ab\dots\dots$ and $mn\dots\dots$ being all positive, and the limits being given by

$$mx+ny+\dots\dots \leqq h.$$

By the general formula

$$E = \frac{1}{\pi}\int_{h_1}^h fu\,du\int_0^\infty d\alpha\int_0^\infty dx\int_0^\infty dy\dots e^{-(max+nby+\dots)}\cos\alpha(mx+ny+\dots-u)$$
$$(h_1 < 0).$$

Let $F = \int_0^\infty dx \int_0^\infty dy \ldots e^{-(max+nby+\ldots)} \cos \alpha\,(mx + ny + \ldots - u)$;

then $F = \cos \alpha u \int_0^\infty dx \int_0^\infty dy \ldots e^{-(max+nby+\ldots)} \cos \alpha\,(mx + ny + \ldots)$

$+ \sin \alpha u \int_0^\infty dx \int_0^\infty dy \ldots e^{-(max+nby+\ldots)} \sin \alpha\,(mx + ny + \ldots)$;

which we may put equal to $G \cos \alpha u + H \sin \alpha u$.

First to find the value of G

$$= \int_0^\infty dx \int_0^\infty dy \ldots e^{-(max+nby+\ldots)} \cos \alpha\,(mx + ny + \ldots).$$

Develope the cosine; the result is composed of terms, each containing sines or cosines of all the variables.

Also by the formulæ

$$\left.\begin{aligned} \int_0^\infty e^{-max} \cos \alpha\, mx\, dx &= \frac{a}{m(a^2 + \alpha^2)} \\ \int_0^\infty e^{-max} \sin \alpha\, mx\, dx &= \frac{\alpha}{m(a^2 + \alpha^2)} \end{aligned}\right\} \ldots\ldots\ldots\ldots(\delta);$$

we see that every factor whether sine or cosine introduces on integration a factor of the form $\dfrac{1}{m(a^2 + \alpha^2)}$. Moreover a sine factor introduces α in the numerator, a cosine factor a or b or &c.

Let P represent the continued product of a, b, c &c. and D that of $m(a^2 + \alpha^2)\ n(b^2 + \alpha^2)$ &c. Then $G = \dfrac{P}{D}\Sigma C \dfrac{\alpha^\mu}{ab\ldots}$, where μ is a positive integer less than the whole number of variables in E, and equal to the number of factors in the denominator of $\dfrac{\alpha^\mu}{ab\ldots}$, and C is some coefficient.

A little consideration shows, that if we develope

$$\cos \alpha \left(\frac{1}{a} + \frac{1}{b} + \ldots\right)$$

in a series of powers and products of

$$\frac{1}{a}\ \frac{1}{b} \ldots\ldots\ldots\ldots,$$

and neglect all terms involving powers above the first of these quantities, the result will be $= \Sigma C \dfrac{\alpha^\mu}{ab\ldots}$.

Consequently

$$\Sigma C \frac{a^{\mu}}{ab\ldots\ldots} = 1 - \alpha^2 \Sigma \frac{1}{a}\frac{1}{b} + \alpha^4 \Sigma \frac{1}{a}\frac{1}{b}\frac{1}{c}\frac{1}{f} - \text{\&c.}$$

$\Sigma \frac{1}{a}\frac{1}{b}$ denoting the combinations two and two of the quantities $\frac{1}{a}, \frac{1}{b} \ldots\ldots$; and so of the rest. For in the developement of $(A + B + \ldots\ldots)^m$ where m is not greater than the number of quantities A, B, &c. there is always a term involving no power above the first of any of these quantities, and its coefficient is $1.2.\ldots.m$. This term is obviously the sum of the combinations m and m together of A, B, &c., and that its coefficient is equal to $1.2.3.\ldots.m$, appears from the polynomial theorem, viz.

$$(A + B + \ldots\ldots)^m = \Sigma \frac{1.2.\ldots.m}{1.2..p.1.2\ldots q\ \text{\&c.}} A^h B^y \ldots\ldots$$

Thus $\quad G = \dfrac{p_t - \alpha^2 p_{t-2} + \alpha^4 p_{t-4} - \text{\&c.}}{mn.\ldots.(a^2 + \alpha^2)(b^2 + \alpha^2).\ldots} \ldots\ldots (4);$

where p_t, p_{t-2} &c. are the alternate coefficients of the equation

$$v^t - p_1 v^{t-1} + \text{\&c.} \quad \pm p_{t-1}\, v \mp p_t = 0,$$

whose roots are a, b, c &c. (I suppose t to be the number of variables in E). In precisely the same way we should find

$$H = \frac{\alpha p_{t-1} - \alpha^3 p_{t-3} + \text{\&c.}}{mn.\ldots.(a^2 + \alpha^2)(b^2 + \alpha^2).\ldots} \ldots\ldots (5).$$

Next to find the values of

$$\int_0^\infty G \cos \alpha u d\alpha \text{ and } \int_0^\infty H \cos \alpha u d\alpha.$$

Let $\quad K = \displaystyle\int_0^\infty \frac{\cos \alpha u d\alpha}{(a^2 + \alpha^2)(b^2 + \alpha^2).\ldots}$

$$= \Sigma \frac{1}{(b^2 - a^2)(c^2 - a^2).\ldots} \int_0^\infty \frac{\cos \alpha u d\alpha}{a^2 + \alpha^2};$$

$$\therefore K = \frac{\pi}{2} \Sigma \frac{1}{a(b^2 - a^2)(c^2 - a^2).\ldots} e^{\mp au} \ldots\ldots (6),$$

the upper sign is to be taken when u is > 0.

By differentiating this for a, we have the value of

$$\int_0^\infty \frac{\alpha \sin \alpha u d\alpha}{(a^2+\alpha^2)(b^2+\alpha^2).\ldots} = \pm \frac{\pi}{2} \Sigma \frac{1}{(b^2-a^2)(c^2-a^2).\ldots} e^{\mp au} \ldots (7),$$

and so by repeated differentiations we find the values of all the integrals which enter into $\int_0^\infty G \cos \alpha u d\alpha$ and $\int_0^\infty H \sin \alpha u d\alpha$.

Thus

$$F=\frac{\pi}{2}\frac{1}{mn\ldots}\,\Sigma\,\{p_t+ap_{t-1}+\&c.+a^t\}\,\frac{e^{-au}}{a\,(b^2-a^2)(c^2-a^2)\ldots}\ \ldots (8),$$

when u is > 0, and

$$F=\frac{\pi}{2}\frac{1}{mn\ldots}\,\Sigma\,\{p_t-ap_{t-1}+\&c.\pm a^t\}\,\frac{e^{+au}}{a\,(b^2-a^2)(c^2-a^2)\ldots}\ \ldots (9),$$

when u is < 0.

But $p_t - ap_{t-1} + \&c. \pm a^t = 0$.

Hence $F = 0$, when u is < 0, and therefore in (E) we may make $h_1 = 0$, so that the limits of integration for u are 0 and h.

Again $\quad a^t + p_1 a^{t-1} + \&c. = (a+a)\,(a+b)\ldots$

and $2a\,(b^2-a^2)\,(c^2-a^2)\ldots = \{2a\,.\,(a+b)\ldots\}\,\{(b-a)(c-a)\ldots\}$.

Therefore

$$\frac{a^t+p_1a^{t-1}+\&c.}{2a\,(b^2-a^2)\,(c^2-a^2)\ldots}=\frac{1}{(b-a)\,(c-a)\ldots}$$

and

$$F=\frac{\pi}{mn\ldots}\,\Sigma\,\frac{e^{-au}}{(b-a)\,(c-a)\ldots}\ \ldots (10),$$

and therefore finally

$$\int_0 dx\int_0 dy\ \ldots\ e^{-(max+nby+\ldots)}f\,(mx+ny+\ldots)$$

$$=\frac{1}{mn\ldots}\,\Sigma\,\frac{\int_0^h fue^{-au}du}{(b-a)\,(c-a)\ldots}\ \ldots\ldots (11).$$

If $a = b = c = \&c. = A$, the first side of this equation

$$=\frac{1}{mn\ldots}\,\frac{1}{\Gamma(t)}\int_0^h fu\,e^{-Au}u^{t-1}du \text{ by } (3).$$

As a verification of our analysis we may remark, that in this case

$$\Sigma\,\frac{e^{-au}}{(b-a)\,(c-a)\ldots}=e^{-Au}\,\frac{u^{t-1}}{\Gamma(t)};$$

for we have, what is probably a known result, and which at any rate may be easily proved

$$\Sigma\,\frac{F(a)}{(b-a)\,(c-a)\ldots}=\frac{1}{\Gamma(t)}\,F^{(t-1)}(A) \text{ when } a=b=\&c.=A,$$

where $F^{(p)}(A)$ denotes the p^{th} derived function of $F(A)$: and this formula applied to the case where $F(a) = e^{-au}$ gives the above written result.

By differentiating (11) for $a, b, c,$ &c. $\lambda, \mu, \nu,$ &c. times respectively, $\lambda, \mu, \nu,$ &c. being integral or fractional, and dividing by $m^\lambda, n^\mu, p^\nu,$ &c. we should obtain the value of

$$\int_0 dx\int_0 dy\ \ldots\ e^{-(max+nby+\ldots)}f\,(mx+ny+\ldots)\,x^\lambda y^\mu\ldots$$

which would include every case of (3). But the investigation would be complex, and I shall therefore only indicate it.

In a future number of the journal I may perhaps apply the method to some other cases, and particularly with regard to such multiple integrals as

$$\int dx \int dy \ldots \phi\,(xy \cdots) f[\psi\,(xy\ \ldots)]\, F[\chi\,(xy\ \ldots)], \text{ \&c.}$$

the limits being given by the series of inequalities,

$$\psi. > h_1 < h,$$
$$\chi. > k_1 < k,$$
$$\text{\&c.} > \text{\&c.} < \text{\&c.}$$

In theory, such an integral is reducible to a multiple integral of as many variables as there are limiting inequalities. But it is not easy to find cases in which this reduction can be actually effected.

II.—ON THE EQUATIONS OF MOTION OF ROTATION.

By Andrew Bell.

THE equations of the motion of rotation of a solid body, acted on by any forces, its centre of inertia being fixed, may be established in a comparatively concise manner by means of one of the theorems contained in the paper at p. 213, vol. III. of this Journal.

Let x', y', z' be the principal axes of the body passing through its centre of inertia, and let p, q, r be the angular velocities about them respectively, then these velocities are the constituents of an angular velocity ω about the instantaneous axis; also let A, B, C be the respective moments of inertia about these axes.

If x', y', z' are the constituents of the impressed forces parallel respectively to these axes, then the moments of these forces about these axes being denoted respectively by N, M, L, and Δm being an element of the body, the moment about the axis (x') is

$$N = \Sigma\,(y'z' - z'y')\,\Delta m.$$

Again, the effective forces in reference to (x') are the moving force about it arising from the constituent rotation around this axis, and also the effect of the centrifugal force arising from the rotation about the instantaneous axis in producing rotation about the axis (x'). Now A is the mass of equivalent inertia with the body, at a distance unity from the axis (x'), p is the absolute velocity at that distance, and therefore $\dfrac{dp}{dt}$

is the accelerating force and $A\,\dfrac{dp}{dt}$ the moving force at the same distance. Also it has been proved, in the third article of the paper referred to above, that the effect of the centrifugal force will be obtained by decomposing the rotation about the instantaneous axis into its constituent rotations about (x') and another axis (z) in the same plane with the two former, and perpendicular to (x'), and then finding the similar effect of the rotation about (z) in producing rotation about (x'). This new axis (z) is evidently the resultant axis of (y') and (z'). If, therefore ω' is the angular velocity around it and (y) an axis perpendicular to the plane of $x'y$, the required force is

$$= \omega'^2\,\Sigma yz\Delta m.$$

This expression will be transformed into one in terms of y', z' by substituting these values

$$y = y'\cos\theta - z'\sin\theta,$$
$$z = y'\sin\theta + z'\cos\theta\,;$$

whence $\omega'^2\Sigma yz\Delta m$

$$= \omega'^2\sin\theta\cos\theta\,\Sigma\,(y'^2 - z'^2)\,\Delta m + \omega'^2\,(\cos^2\theta - \sin^2\theta)\,\Sigma y'z'\Delta m.$$

But for the principal axes $\Sigma y'z'\Delta m = 0$,

also $\quad \Sigma(y'^2 - z'^2)\,\Delta m = \Sigma\{(x'^2 + y'^2) - (x'^2 + z'^2)\}\,\Delta m = C - B\,;$

and $\quad q = \omega'\sin\theta, \quad r = \omega'\cos\theta\,;$

and therefore $\quad \omega'^2\Sigma yz\Delta m = (C - B)\,qr.$

Hence the moment of the effective forces in reference to the axis (x') is

$$= A\,\frac{dp}{dt} + (C - B)\,qr.$$

As the impressed and effective forces must equilibrate, therefore

$$A\,\frac{dp}{dt} + (C - B)\,qr - N = 0.$$

In the same manner the other two equations of the motion of rotation about the axes (y'), (z') are found to be

$$B\,\frac{dq}{dt} + (A - C)\,pr - M = 0,$$

$$C\,\frac{dr}{dt} + (B - A)\,qr - L = 0.$$

In considering the terms of any of these equations, as for instance the first, it is evident that N being the only impressed force, the other two terms must express effective forces, and the first term evidently referring to the moving force about the axis (x'), the second term must represent the

effect of centrifugal force. This has been remarked by a writer on this subject, (*Westminster Review*, vol. II. p. 321), but he has given no proof of it independently of the common demonstration. This is the only troublesome term to investigate, and to establish its origin on a new principle the theorem referred to in the first paragraph of this paper has here been employed.

III.—ON A METHOD OF FINDING THE GREATEST COMMON MEASURE OF TWO POLYNOMIALS.

By A. Q. G. CRAUFURD, M.A. Jesus College.

To find the greatest common measure of two rational and entire functions of x, X_1 and X_2.

Assume y to be one of the roots of the greatest common measure, then y must also be a root of each of the proposed polynomials X_1 and X_2, consequently it satisfies the equations

$$X_1 = 0, \quad X_2 = 0 \ldots\ldots\ldots\ldots (1).$$

The converse of this proposition is also true; viz. "whatever satisfies both these equations is a root of the greatest common measure."

To prove this, let the greatest common measure be denoted by M, and let

$$X_1 = M.Q_1, \quad X_2 = M.Q_2.$$

Then whatever is a root of X_1, must be a root of M or of Q_1; and whatever is a root of X_2 must be a root of M or of Q_2; therefore every root of X_1 *and* X_2 is a root of M or of Q_1 *and* Q_2. But Q_1 and Q_2 have no common measure, and therefore no common root; therefore, every root of X_1 *and* X_2 is a root of M. It is proved, then, that the roots common to X_1 and X_2 are roots, and the *only* roots, of M.

The problem is thus reduced to that of determining an equation which shall have all the roots which are *common* to the two equations (1), and no other roots. This is easily done by the method employed in my paper on "Elimination," in the XIIth number of this *Journal*. I will first explain the process in general terms, and then apply it to an example; and in treating the general case I shall first suppose X_1 and X_2 to be both of the n^{th} degree.

If equations (1) have all their roots in common, either of them is the equation sought, and what is required is already done. If these equations do not coincide y must have less than n values.

Take A_n, A_{n-1}, A_{n-2}. . . A_0 to represent the coefficients of the powers of x in X_1, and let B_n, B_{n-1}, B_{n-2}. . . B_0, be the corresponding coefficients in X_2.
Let p and q be two multipliers, such that

$$pA_n - qB_n = 0.$$

Then the equation $\quad pX_1 - qX_2 = 0$

is of the $(n-1)^{\text{th}}$ degree. Let r and s be such multipliers that

$$rA_0 - sB_0 = 0;$$

then the equation $\quad rX_1 - sX_2 = 0$

is of the n^{th} degree, but it is *deprived of its last term.*

It is evident that whatever satisfies equations (1) must also satisfy these. It is likewise true that whatever satisfies these must also satisfy equations (1); (except in a particular case which will be specially noticed presently). For, if we multiply the first of the two derived equations by s and the second by q, and subtract, we have

$$(ps - rq)\ X_1 = 0.$$

Again, if we multiply the first of the two derived equations by r and the second by p, and subtract, we have

$$(ps - rq)\ X_2 = 0.$$

Therefore (if $ps - rq$ is not equal to zero), the two derived equations are convertible with the two proposed. The second of the two derived equations has zero for a root, but the other has not, therefore this root is not common to the two derived nor to the two proposed; we may therefore reject it, and the two derived equations will still contain all the roots *common* to the two proposed, and they will both be of the $(n-1)^{\text{th}}$ degree.

Let $pX_1 - qX_2$ and $\dfrac{rX_1 - sX_2}{x}$ be denoted by X_1' and X_2' respectively, then $\quad X_1' = 0$ and $X_2' = 0$ . . . . (2)

are equations of the $(n-1)^{\text{th}}$ degree which are convertible with the proposed.

If these equations have all their roots in common, either of them is the equation sought. If not y can not have $(n-1)$ values. Form, therefore, from equations (2), two equations of the $(n-2)^{\text{th}}$ degree, in the same manner that equations (2) were formed from equations (1). If these coincide, and have all their roots in common, either of them is the equation sought. If not, y can not have $(n-2)$ values, and the process of reduction must be continued, and if this process leads at

last to a pair of equations which *do coincide*, and *have* all their roots in common, either of this pair of equations is the equation sought.

Let $X^p = 0$ be this equation. Then, since X^p is a function which has all the roots of the greatest common measure, and no others, it must be that common measure.

If the process of reduction never leads to a pair of equations which have their roots in common, the two proposed functions will, *by that process*, have been proved to have no common measure.

If the two proposed functions are not of the same degree, suppose X_1 to be of the m^{th} and X_2 of the n^{th} degree, m being $> n$. Assume, as before, $A_m, A_{m-1}, A_{m-2} \ldots A_0$ to be the coefficients of the powers of x in X_1 and $B_n, B_{n-1}, B_{n-2} \ldots B_0$ the corresponding coefficients in X_2. Let p and q be such multipliers that

$$pA_m - qB_n = 0.$$

Then the equation $\quad pX_1 - qx^{m-n}X_2 = 0$

is of the $(n-1)^{\text{th}}$ degree, and this equation, together with the second of the proposed (viz. $X_2 = 0$), are convertible with the two proposed.

If we call these $\quad X_1' = 0, \quad X_2 = 0 \ldots\ldots\ldots\ldots (2),$

we may continue the process till we obtain two of the n^{th} degree, and then proceed as before.

It remains to notice the particular case in which the multipliers p, s, q, and r, are such that $ps - rq = 0$.

In this case we must proceed in the reduction by a different method. First form, as before, the equation

$$pX_1 - qX_2 = 0, \quad \text{or} \quad X_1' = 0.$$

This, together with $X_1 = 0$, is evidently convertible with the proposed. Let B'_{n-1}, B'_{n-2}, &c. denote the coefficients of X_1'. And let p' and q' be such that $p'A_n - q'B'_{n-1} = 0$. Then the equation $p'X_1 - q'xX'_1 = 0$ is of the $(n-1)^{\text{th}}$ degree; and the two equations $\quad X_1' = 0, \quad pX_1 - q'xX_1' = 0$

are evidently convertible with

$$X_1 = 0, \text{ and } X_1' = 0;$$

therefore they are convertible with the proposed.

This latter method of reduction may always be employed, if preferred, instead of that previously given.

Ex. Let the functions be

$$2x^4 - 12x^3 + 19x^2 - 6x + 9,$$

and

$$8x^3 - 36x^2 + 38x - 6.$$

Write $2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0$
and $4x^3 - 18x^2 + 19x - 3 = 0$ .... (1).

Multiply the first of these equations by 2, and the second by x, and subtract, and you have

$$-6x^3 + 19x^2 - 9x + 18 = 0.$$

Consequently the two equations of the third degree are

$6x^3 - 19x^2 + 9x - 18 = 0$
and $4x^3 - 18x^2 + 19x - 3 = 0$ .... (2).

Multiply the first of these by 2, and the second by 3, and subtract: you obtain

$$16x^2 - 39x - 27 = 0.$$

Multiply the first of equations (2) by 1, and the second by 6, subtract and divide by x, you obtain

$$18x^2 - 89x + 105 = 0.$$

So that the two equations of the second degree are

$16x^2 - 39x - 27 = 0$
and $18x^2 - 89x + 105 = 0$ .... (3).

Multiply the first of these by 9, and the second by 8, and subtract: you obtain $361x - 1083 = 0$,

$$\text{or} \quad x - 3 = 0.$$

Multiply the first of equations (3) by 35, and the second by 9, add, and divide by x: you obtain

$$722x - 2166 = 0,$$

$$\text{or} \quad x - 3 = 0.$$

The two equations of the first degree *coincide*, and $x - 3$ is the greatest common measure of the proposed functions.

The process of reduction might have been performed in the following manner, which, though differing slightly in the steps, leads of course to the same result.

Take the two equations of the second degree, viz.

$6x^3 - 19x^2 + 9x - 18 = 0$
and $4x^3 - 18x^2 + 19x - 3 = 0$ .... (2).

Form, as before, the equation

$$16x^2 - 39x - 27 = 0.$$

Multiply the first of equations (2) by 8, and the equation just formed by $3x$, and subtract: you obtain

$$35x^2 - 153x + 144 = 0.$$

So that the two equations of the second degree are

$$\left.\begin{array}{r} 16x^2 - 39x - 27 = 0 \\ \text{and} \qquad 35x^2 - 153x + 144 = 0 \end{array}\right\} \ldots\ldots\ldots\ldots (3).$$

Multiply the first of these by 35, and the second by 16, and subtract. The result is

$$1083x - 3249 = 0,$$

$$\text{or} \quad x - 3 = 0.$$

Multiply this by $16x$, and subtract from the first of equations (3): you obtain

$$9x - 27 = 0,$$

$$\text{or} \quad x - 3 = 0.$$

As before, we find that the two equations of the first degree *coincide*, and $x - 3$ is the greatest common measure. The latter method is rather shorter than the former.

IV.—ON THE SYMMETRICAL INVESTIGATION OF POINTS OF INFLECTION.

By W. WALTON, M.A. Trinity College.

THE condition ordinarily adopted for the discovery of a point of inflection is, that $\frac{d^2y}{dx^2}$ shall change sign as x passes through the value which it has at the point; whence it follows that at the point itself $\frac{d^2y}{dx^2} = 0$, or $= \infty$. This method of investigation is sufficiently convenient when we have given y explicitly in a rational function of x; when, however, x and y are involved implicitly it frequently gives rise to painful, and at all times to inelegant, operations. The method which we lay before the readers of the *Journal* has the advantage of symmetry, and is, in many cases, especially when x and y are symmetrically involved in the equation to the curve, free from vexatious processes. The intelligence of the Algebraist will enable him to see in particular cases which method ought to be chosen. The symmetrical method is often very useful when the branches of a multiple point have inflection at the point.

Let the equation to an algebraical curve, cleared of radicals and negative indices, be represented by

$$F = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where F is a rational function of x and y.

Let ds denote an element of the arc of the curve at the point x, y, and let s be taken as the independent variable.

Let
$$U = \frac{dF}{dx}, \quad V = \frac{dF}{dy},$$
$$u = \frac{d^2F}{dx^2}, \quad w = \frac{d^2F}{dxdy}, \quad v = \frac{d^2F}{dy^2},$$
$$l = \frac{dx}{ds}, \quad m = \frac{dy}{ds}, \quad l' = \frac{dl}{ds}, \quad m' = \frac{dm}{ds}.$$

Then, differentiating (1), we have
$$lU + mV = 0 \ldots\ldots\ldots\ldots\ldots (2);$$
differentiating (2),
$$l^2u + 2lmw + m^2v + l'U + m'V = 0 \ldots\ldots (3).$$
Again, it is clear that $l^2 + m^2 = 1$, and therefore
$$ll' + mm' = 0 \ldots\ldots\ldots\ldots\ldots (4).$$
From (2) and (4) we get
$$l'V = m'U;$$
hence, multiplying (3) by U, we obtain
$$U(l^2u + 2lmw + m^2v) + l'(U^2 + V^2) = 0,$$
or, by virtue of (2),
$$\frac{l^2U}{V^2}(uV^2 - 2UVw + U^2v) + l'(U^2 + V^2) = 0 \ldots (5).$$
In a similar way we may shew that
$$\frac{m^2V}{U^2}(uV^2 - 2UVw + U^2v) + m'(U^2 + V^2) = 0 \ldots (6).$$
Again, by the relation $l^2 + m^2 = 1$ and the equation (2), we get
$$\frac{l^2}{V^2} = \frac{1}{U^2 + V^2} = \frac{m^2}{U^2}:$$
hence (5) and (6) give us
$$U(uV^2 - 2UVw + U^2v) + l'(U^2 + V^2)^2 = 0 \ldots (7),$$
$$V(uV^2 - 2UVw + U^2v) + m'(U^2 + V^2)^2 = 0 \ldots (8).$$
Now at a point of inflection it is evident that l and m must be the one a maximum and the other a minimum: hence, as we pass in the neighbourhood of the point along the curve from one side of the point to the other, we know by the theory of maxima and minima that l' and m' must each of them change sign. It is evident, then, from (7) and (8), that
$$U(uV^2 - 2UVw + U^2v)$$

and $$V(uV^2 - 2UVw + U^2v)$$
must both change sign.

Suppose first that neither U nor V changes sign as we pass through the point; then the sufficient and necessary condition for a change of sign in the value of l' and m', is that

$$uV^2 - 2UVw + U^2v \text{ . . . . } (9)$$

change sign as we pass through the point; this condition evidently involves the fact, that at the point itself

$$uV^2 - 2UVw + U^2v = 0 \text{ . . . . } (10).$$

Next suppose that U changes sign; then it is evident that the expression (9) must not change sign, for otherwise l', as will be evident from the formula (7), could not change sign. But from (8) we see that for a change of sign in the value of m', one and one only of the quantities V and (9) must change sign; hence V only must change sign. Thus we see that if either of the quantities U and V change sign, both must do so, and that (9) must not change sign. If U and V both change sign it is clear that at the point itself

$$U = 0, \quad V = 0,$$

which are the conditions for multiplicity of branches at the point.

The general rule, therefore, for finding points of inflection may be thus enunciated. First ascertain the values of x and y which will satisfy simultaneously the equations

$$F = 0, \quad uV^2 - 2UVw + vU^2 = 0,$$

and reject all the pairs of values thus obtained which do not, as we pass through the corresponding points, correspond to a change of sign in the expression

$$uV^2 - 2UVw + vU^2,$$

or which do correspond to a change of sign in either U or V; the pairs of values of x and y, which are retained, correspond to points of inflection.

Secondly, ascertain those pairs of values which satisfy simultaneously $F = 0, \quad U = 0, \quad V = 0,$

and reject all of these pairs which do not correspond to a change of sign in both U and V as we pass through the corresponding points along one or other of the branches, or which do correspond to a change of sign in the expression

$$uV^2 - 2UVw + vU^2.$$

In the preceding investigation we have supposed F to be a rational function of x and y. Should this not be the case it

will be evident, from what has been said, that in addition to the values of x and y, which may be obtained by the rule which we have enunciated, we must likewise take those which will render in the first case,

$$F = 0, \quad uV^2 - 2UVw + vU^2 = \infty;$$

and in the second case,

$$F = 0, \quad U = \infty, \quad V = \infty;$$

the conditions depending on change of sign being the same as before.

Ex. 1. Let the curve be

$$F = ax^3 + by^3 - c^4 = 0.$$

Then

$$U = 3ax^2, \quad V = 3by^2,$$

$$u = 6ax, \quad w = 0, \quad v = 6by.$$

Hence, from the formula (10) there is, if we cast out constant factors,

$$xy\,(ax^3 + by^3) = 0;$$

or, by the equation to the curve,

$$xy = 0.$$

Thus $x = 0$, or $y = 0$, and in both cases neither U nor V changes sign, while the formula (9) does change sign. Hence we have two points of inflection

$$x = 0, \quad y = \frac{c^{\frac{4}{3}}}{b^{\frac{1}{3}}},$$

and

$$x = \frac{c^{\frac{4}{3}}}{a^{\frac{1}{3}}}, \quad y = 0.$$

Ex. 2. Take the curve

$$F = (x^2 + y^2)^2 - a^2x^2 + b^2y^2,$$

and suppose that we wish to find whether there be a point of inflection at the origin. Then

$$U = 2x\,(2x^2 + 2y^2 - a^2),$$

$$V = 2y\,(2x^2 + 2y^2 + b^2),$$

$$u = 12x^2 + 4y^2 - 2a^2,$$

$$v = 4x^2 + 12y^2 + 2b^2,$$

$$w = 8xy.$$

From these results it is evident that U and V both change sign if we change x and y each of them from ± 0 to ∓ 0.

Moreover it is clear that neither uV^2, vU^2, nor $2UVw$, experience any change of sign when we put $\pm x$, $\pm y$ for $\mp x$, $\mp y$ respectively. Hence the expression (9) does not change sign. If we had kept y positive or negative throughout while we changed x from ± 0 to ∓ 0, the expression (9) would have changed sign, and flexure would not have taken place. Hence we see that the branch which passes through the origin from below to above the axis of x, or that which passes from above to below, will have an inflection at the origin. The discussion of this example by the unsymmetrical method would have been much more embarrassing.

Ex. 3. Let the curve be

$$F = \left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}} - 1 = 0.$$

Then $$U = \frac{1}{3a}\left(\frac{x}{a}\right)^{-\frac{2}{3}}, \quad V = \frac{1}{3b}\left(\frac{y}{b}\right)^{-\frac{2}{3}},$$

$$u = -\frac{2}{9a^2}\left(\frac{x}{a}\right)^{-\frac{5}{3}}, \quad w = 0, \quad v = -\frac{2}{9b^2}\left(\frac{y}{b}\right)^{-\frac{5}{3}}.$$

Hence the expression

$$uV^2 - 2UVw + vU^2$$

will vary as $$\left(\frac{x}{a}\right)^{-\frac{5}{3}}\left(\frac{y}{b}\right)^{-\frac{4}{3}} + \left(\frac{y}{b}\right)^{-\frac{5}{3}}\left(\frac{x}{a}\right)^{-\frac{4}{3}}$$

$$= \left(\frac{x}{a}\right)^{-\frac{5}{3}}\left(\frac{y}{b}\right)^{-\frac{5}{3}}\left\{\left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}}\right\}$$

$$= \left(\frac{x}{a}\right)^{-\frac{5}{3}}\left(\frac{y}{b}\right)^{-\frac{5}{3}}.$$

Putting this expression $= \infty$, we get $x = 0$, or $y = 0$; and therefore by the equation to the curve $y = b$, $x = a$, respectively. It is evident then that as x passes through 0, U and V do not change sign while the expression (9) does: and similarly for y; hence there are two points of inflection, viz. $x = 0$, $y = b$, and $x = a$, $y = 0$.

V.—DEMONSTRATION OF PASCAL'S THEOREM.

By A. CAYLEY, B.A. Fellow of Trinity College.

LEMMA 1. Let $U = Ax + By + Cz = 0$ be the equation to a plane passing through a given point taken for the origin, and consider the planes

$$U_1 = 0,\quad U_2 = 0,\quad U_3 = 0,\quad U_4 = 0,\quad U_5 = 0,\quad U_6 = 0.$$

The condition which expresses that the intersections of the planes (1) and (2), (3) and (4), (5) and (6) lie in the same plane, may be written down under the form

$$\begin{vmatrix} A_1 & A_2 & A_3 & A_4 & . & . \\ B_1 & B_2 & B_3 & B_4 & . & . \\ C_1 & C_2 & C_3 & C_4 & . & . \\ . & . & A_3 & A_4 & A_5 & A_6 \\ . & . & B_3 & B_4 & B_5 & B_6 \\ . & . & C_3 & C_4 & C_5 & C_6 \end{vmatrix} = 0.$$

LEMMA 2. Representing the determinants

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \text{ \&c.}$$

by the abbreviated notation $\overline{123}$, &c. The following equation is identically true:

$$\overline{345}\,.\,\overline{126} - \overline{346}\,.\,\overline{125} + \overline{356}\,.\,\overline{124} - \overline{456}\,.\,\overline{123} = 0.$$

This is an immediate consequence of the equations

$$\begin{vmatrix} . & . & x_3 & x_4 & x_5 & x_6 \\ . & . & y_3 & y_4 & y_5 & y_6 \\ . & . & z_3 & z_4 & z_5 & z_6 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{vmatrix} = \begin{vmatrix} . & . & x_3 & x_4 & x_5 & x_6 \\ . & . & y_3 & y_4 & y_5 & y_6 \\ . & . & z_3 & z_4 & z_5 & z_6 \\ x_1 & x_2 & . & . & . & . \\ y_1 & y_2 & . & . & . & . \\ z_1 & z_2 & . & . & . & . \end{vmatrix} = 0.$$

Consider now the points 1, 2, 3, 4, 5, 6, the co-ordinates of these being respectively $x_1, y_1, z_1 \ldots x_6, y_6, z_6$. I represent, for shortness, the equation to the plane passing through the origin, and the points 1, 2, which may be called the plane $\overline{12}$, in the form

$$x\,.\,\overline{12}_x + y\,.\,\overline{12}_y + z\,.\,\overline{12}_z = 0.$$

Consequently the symbols $\overline{12}_x, \overline{12}_y, \overline{12}_z$ denote respectively $y_1z_2 - y_2z_1,\ z_1x_2 - z_2x_1,\ x_1y_2 - x_2y_1$, and similarly for the planes $\overline{13}$, &c. If now the intersections of $\overline{12}$ and $\overline{45}$, $\overline{23}$ and $\overline{56}$,

$\overline{34}$ and $\overline{61}$ lie in the same plane, we must have, by Lemma (1), the equation

$$\begin{vmatrix} 12_x & 45_x & 23_x & 56_x & . & . \\ 12_y & 45_y & 23_y & 56_y & . & . \\ 12_z & 45_z & 23_z & 56_z & . & . \\ . & . & 23_x & 56_x & 34_x & 61_x \\ . & . & 23_y & 56_y & 34_y & 61_y \\ . & . & 23_z & 56_z & 34_z & 61_z \end{vmatrix} = 0.$$

Multiplying the two sides of this equation by the two sides respectively of the equation

$$\begin{vmatrix} x_6 & x_1 & x_2 & . & . & . \\ y_6 & y_1 & y_2 & . & . & . \\ z_6 & z_1 & z_2 & . & . & . \\ . & . & . & x_3 & x_4 & x_5 \\ . & . & . & y_3 & y_4 & y_5 \\ . & . & . & z_3 & z_4 & z_5 \end{vmatrix} = \overline{612}\,.\,\overline{345}.$$

And observing the equations

$$x_6\overline{12}_x + y_6\overline{12}_y + z_6\overline{12}_z = \overline{612},\ \ \overline{112} = 0,\ \&c.$$

This becomes

$$\begin{vmatrix} 612 & . & . & . & . & . \\ 645 & 145 & 245 & . & . & . \\ 623 & 123 & . & . & 423 & 523 \\ . & 156 & 256 & 356 & 456 & . \\ . & . & . & 361 & 461 & 561 \end{vmatrix} = 0.$$

Reducible to

$$\overline{612}\ \overline{534}\begin{vmatrix} \overline{145} & \overline{245} & . & . \\ 123 & . & . & 423 \\ 156 & 256 & 356 & 456 \\ . & . & 361 & 461 \end{vmatrix} = 0.$$

Or, omitting the factor $\overline{612}\ \overline{534}$ and expanding,

$$\overline{145}\,.\,\overline{256}\,.\,\overline{423}\,.\,\overline{361} + \overline{245}\,.\,\overline{123}\,.\,\overline{456}\,.\,\overline{361}$$
$$- \overline{245}\,.\,\overline{123}\,.\,\overline{356}\,.\,\overline{461} - \overline{245}\,.\,\overline{156}\,.\,\overline{423}\,.\,\overline{361} = 0.$$

Considering for instance x_6, y_6, z_6 as variable, this equation expresses evidently that the point (6) lies in a cone of the second order having the origin for its vertex, and the equation is evidently satisfied by writing $x_6, y_6, z_6 = x_1, y_1, z_1$ or x_3, y_3, z_3 or x_4, y_4, z_4 or x_5, y_5, z_5, or the cone passes through the

points 1, 3, 4, 5. For $x_6, y_6, z_6 = x_2, y_2, z_2$, the equation becomes, reducing and dividing by $\overline{245}\ \overline{123}$,

$$\overline{452}\,.\,\overline{321} - \overline{352}\,.\,\overline{421} + \overline{152}\,.\,\overline{423} = 0,$$

which is deducible from Lemma (2), by writing $x_6, y_6, z_6 = x_2, y_2, z_2$, and is therefore identically true. Hence the cone passes through the point (2), and therefore the points 1, 2, 3, 4, 5, 6 lie in the same cone of the second order, which is Pascal's Theorem. I have demonstrated it in the cone, for the sake of symmetry; but by writing throughout unity instead of z, the above applies directly to the case of the theorem in the plane.

The demonstration of Chasles' form of Pascal's Theorem (viz. that the anharmonic relation of the planes $\overline{61}, \overline{62}, \overline{63}, \overline{64}$ is the same with that of $\overline{51}, \overline{52}, \overline{53}, \overline{54}$), is very much simpler; but as it would require some preparatory information with reference to the analytical definition of the similarity of anharmonic relation, I must defer it to another opportunity.

VI.—ON THE TRANSFORMATION OF MULTIPLE INTEGRALS.

By George Boole.

The result of the elimination of the variables from the equations

$$\left.\begin{array}{l} a_1x_1 + a_2x_2 \ldots + a_nx_n = 0 \\ b_1x_1 + b_2x_2 \ldots + b_nx_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ r_1x_1 + r_2x_2 \ldots + r_nx_n = 0 \end{array}\right\} \ldots\ldots\ldots (1)$$

is an equation of which the second member is 0, and of which the first member is formed from the coefficient of $x_1x_2..x_n$ in the product of the given equations, by assuming a particular term, as $a_1b_2..r_n$, positive, and applying to every other term a change of sign, for every binary permutation of the suffixes which it may exhibit, when compared with the proposed term, $a_1b_2..r_n$.

By a binary permutation is here meant a permutation affecting only the relative position of two quantities, as 123, 132; or 123, 213.

The truth of the above theorem is shewn by the following considerations. The elimination of x_1 from the first and second equation of the system (1) introduces terms of the form $a_1b_2 - a_2b_1$, $a_1b_3 - a_3b_1$, &c., in which the law of binary permutation is apparent, and as we may begin the process of elimination with any variable and with any pair of equations,

the law is universal. From the same instance it is evident that no proposed suffix can occur twice in a given term, which condition is also characteristic of the coefficient of $x_1x_2..x_n$ in the product of the equations of the system, whence the theorem is manifest.

As an example let $n = 4$, and assuming the term $a_1b_2c_3d_4$ positive, let it be required to find the signs of the terms $a_1b_3c_2d_4$, $a_4b_2c_1d_3$, $a_3b_4c_2d_1$. We proceed thus:

1324	4213	3421
1234	1243	1423
	1234	1243
		1234

Here the term $a_1b_3c_2d_4$ is reduced to $a_1b_2c_3d_4$ by a single binary permutation of the suffixes, viz. 32 to 23, wherefore the sign is negative; the term $a_4b_2c_1d_3$ undergoing two permutations is positive, and the term $a_3b_4c_2d_1$ undergoing three is negative.

The above theorem may be conveniently applied when we wish to ascertain the result of elimination from a system of equations, to which, from their number, it might be difficult to apply a general form. If, for example, we had the system

$$\left.\begin{array}{r} a_1x_1 = 0 \\ b_1x_1 + b_2x_2 = 0 \\ c_1x_1 + c_2x_2 + c_3x_3 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ r_1x_1 + r_2x_2..+ r_nx_n = 0 \end{array}\right\} \cdots\cdots\cdots\cdots (2),$$

our theorem would at once give

$$a_1b_2c_3..r_n = 0,$$

a result to which we shall have occasion to refer.

In what follows we shall employ the term "final derivative" to the first member of the equation expressing the final result of elimination from a proposed system of equations.

Now let it be required to transform the multiple integral $\int\int..dx_1dx_2..dx_n$ into one depending on the variables $u_1u_2..u_n$, by virtue of the equations $x_1 = U_1$, $x_2 = U_2$, $x_n = U_n$. Differentiating, we have

$$\left.\begin{array}{l} dx_1 = \dfrac{dU_1}{du_1}du_1 + \dfrac{dU_1}{du_2}du_2..+ \dfrac{dU_1}{du_n}du_n \\ dx_2 = \dfrac{dU_2}{du_1}du_1 + \dfrac{dU_2}{du_2}du_2..+ \dfrac{dU_2}{du_n}du_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ dx_n = \dfrac{dU_n}{du_1}du_1 + \dfrac{dU_n}{du_2}du_2..+ \dfrac{dU_n}{du_n}du_n \end{array}\right\} \cdots\cdots (3).$$

Now when x_1 varies, $x_2, x_3 \ldots x_n$ are constant, and $dx_2 \ldots dx_n$ vanish. Hence, if for simplicity we write $\frac{dU_1}{du_1} = a_1$, $\frac{dU_2}{du_1} = b_1$, &c., we have, on the above condition,

$$\left.\begin{array}{r} dx_1 = a_1 du_1 + a_2 du_2 \ldots + a_n du_n \\ 0 = b_1 du_1 + b_2 du_2 \ldots + b_n du_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = r_1 du_1 + r_2 du_2 \ldots + r_n du_n \end{array}\right\} \ldots\ldots (4),$$

whence a relation may be found connecting x_1 and u_1. Let $v_1 = 1$, then the above system of equations may be thus written:

$$\left.\begin{array}{r} (a_1 du_1 - dx_1)\, v_1 + a_2 du_2 \ldots + a_n du_n = 0 \\ (b_1 du_1)\, v_1 + b_2 du_2 \ldots + b_n du_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (r_1 du_1)\, v_1 + r_2 du_2 \ldots + r_n du_n = 0 \end{array}\right\} \ldots (5).$$

The result of the elimination of $v_1, du_2 \ldots du_n$ from these equations, being linear with respect to the coefficients of v_1, will be of the form

$$L dx_1 + M du_1 = 0 \;\ldots\ldots\ldots\ldots (6).$$

In the system (5) let $dx_1 = 0$, then the result of elimination is evidently

$$E_1 du_1 = 0 \;\ldots\ldots\ldots\ldots\ldots (7),$$

where E_1 is the final derivative of the second members of (4) equated to 0, whence $M = E_1$. Again in (5) let $du_1 = 0$, we have

$$\begin{array}{r} (-\,dx)\, v_1 + a_2 du_2 \ldots + a_n du_n = 0, \\ b_2 du_2 \ldots + b_n du_n = 0, \\ \cdots\cdots\cdots\cdots\cdots \\ r_2 du_2 \ldots + r_n du_n = 0\,; \end{array}$$

whence by the theorem, as the result of elimination,

$$-\,E_2 dx = 0,$$

where E_2 is the final derivative of the system

$$\begin{array}{r} b_2 du_2 + b_3 du_3 \ldots + b_n du_n = 0, \\ c_2 du_2 + c_3 du_3 \ldots + c_n du_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ r_2 du_2 + r_3 du_3 \ldots + r_n du_n = 0\,; \end{array}$$

wherefore $L = -\,E_2$, and

$$-\,E_2 dx_1 + E_1 du_1 = 0,$$

$$dx_1 = \frac{E_1}{E_2}\, du_1 \ldots\ldots\ldots\ldots\ldots (8).$$

Now, resuming (3), let x_2 vary, then $dx_1 dx_3 . . dx_n$ vanish, wherefore by (8), $du_1 = 0$, and we have

$$\left.\begin{aligned} dx_2 &= b_2 du_2 + b_3 du_3 . . + b_n du_n \\ 0 &= c_2 du_2 + c_3 du_3 . . + c_n du_n \\ 0 &= r_2 du_2 + r_3 du_3 . . + r_n du_n \end{aligned}\right\} \dots\dots\dots (9).$$

Proceeding with this system of equations as with (4), we get

$$dx_2 = \frac{E_2}{E_3} du_2 \dots\dots\dots\dots\dots (10),$$

where E_3 is the final derivative of the system

$$c_3 du_3 + c_4 du_4 . . + c_n du_n = 0,$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$r_3 du_3 + r_4 du_4 . . + r_n du_n = 0 ;$$

finally we have $$dx_n = E_n du_n \dots\dots\dots\dots\dots (11);$$

wherein $E_n = r_n$. Now multiply (8), (10), . . (11) together, and there results

$$dx_1 dx_2 . . dx_n = E_1 du_1 du_2 . . du_n \dots\dots\dots (12).$$

The sign of E_1 is ambiguous, for it will depend on the order in which du_1, du_2, . . du_n are eliminated,—a thing in its nature indifferent.

Now to extend this theorem to the case in which the two sets of variables are indiscriminately mixed, let $V_1 = 0$, $V_2 = 0$, $V_n = 0$ be the proposed equations. Differentiating we have

$$\left.\begin{aligned} \frac{dV_1}{dx_1} dx_1 . . + \frac{dV_1}{dx_n} dx_n &= - \frac{dV_1}{du_1} du_1 . . - \frac{dV_1}{du_n} du_n \\ \frac{dV_n}{dx_1} dx_1 . . + \frac{dV_n}{dx_n} dx_n &= - \frac{dV_n}{du_1} du_1 . . - \frac{dV_n}{du_n} du_n \end{aligned}\right\}$$

Let E be the final derivative of the first members equated to 0, E' that of the second members equated to 0, then $\frac{E'}{E} = E_1$, by the theory of linear transformations (*Journal*, vol. III. p. 109), whence

$$dx_1 dx_2 . . dx_n = \frac{E'}{E} du_1 du_2 . . du_n \dots\dots (13).$$

On account of the ambiguous signs of E and E', it is not even necessary that the terms involving $du_1\ du_2 . . du_n$ should be transposed to the second members. The general result of our investigations may therefore be expressed in the following Rule.

To transform any multiple integral, $\iint..Xdx_1dx_2..dx_n$, into one depending on $u_1, u_2, ..u_n$, having given the equations connecting the two sets of variables.

Rule. *Differentiate the given equations relatively to $x_1, x_2..x_n$, and eliminating $dx_1dx_2..dx_n$, let E be the final derivative. Proceed in the same manner with $u_1, u_2..u_n$, and let E' be the corresponding final derivative, then*

$$dx_1dx_2..dx_n = \frac{E'}{E}\,du_1du_2..du_n \ldots\ldots (14).$$

Ex. 1. Let the equations of transformation be

$$\begin{aligned} x_1 &= r\cos\theta_1, \\ x_2 &= r\sin\theta_1\cos\theta_2, \\ x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3, \\ x_{n-1} &= r\sin\theta_1\sin\theta_2..\sin\theta_{n-2}\cos\theta_{n-1}, \\ x_n &= r\sin\theta_1\sin\theta_2..\sin\theta_{n-1}. \end{aligned}$$

Squaring and adding, we have

$$x_1^2 + x_2^2..+x_n^2 = r^2,$$

which we shall employ in place of the last equation of the system. Now, differentiating with respect to $x_1x_2..x_n$, we get

$$\begin{aligned} dx_1 &= 0, \\ dx_2 &= 0, \\ 2x_1dx_1..+2x_ndx_n &= 0; \end{aligned}$$

whence, by (2), $E = 2x_n$. Again, differentiating with respect to $r, \theta_1,..\theta_{n-1}$, we have

$$\begin{aligned} 2rdr &= 0, \\ \cos\theta_1 dr - r\sin\theta_1 d\theta_1 &= 0, \\ ..-r\sin\theta_1\sin\theta_2..\sin\theta_{n-1}\,d\theta_{n-1} &= 0; \end{aligned}$$

whence $E' = (-)^{n-1}\,2r^n(\sin\theta_1)^{n-1}(\sin\theta_2)^{n-2}..\sin\theta_{n-1}$, and

$$\begin{aligned} dx_1dx_2..dx_n &= \pm\frac{r^n(\sin\theta_1)^{n-1}(\sin\theta_2)^{n-2}..\sin\theta_{n-1}}{x_n}\,drd\theta_1..d\theta_{n-1} \\ &= \pm\, r^{n-1}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}..\sin\theta_{n-2}\,d\theta_1d\theta_2..d\theta_{n-1}. \end{aligned}$$

If $n = 3$, the above is the transformation from rectangular to polar co-ordinates.

Ex. 2. Let the equations of transformation be

$$\begin{aligned} x_1 &= 1 - u_1, \\ x_2 &= u_1(1-u_2), \\ x_3 &= u_1u_2(1-u_3), \\ x_n &= u_1u_2..u_{n-1}(1-u_n). \end{aligned}$$

Here $E = 0$, and differentiating with respect to $u_1, u_2, .. u_n$,

$$
\begin{aligned}
- du_1 &= 0,\\
(1 - u_2)\, du_1 - u_1 du_2 &= 0,\\
\ldots\ldots - u_1 u_2 du_3 &= 0,\\
\ldots\ldots - u_1 u_2 .. u_{n-1}\, du_n &= 0\,;
\end{aligned}
$$

whence $E' = (-)^n u_1^{n-1} u_2^{n-2} .. u_{n-1}$, wherefore

$$dx_1 dx_2 .. dx_n = \pm\, u_1^{n-1} u_2^{n-2} .. u_{n-1} du_1 du_2 .. du_n \ldots (15).$$

The above transformation leads to a very elegant proof of M. Liouville's theorem expressing the value of the definite multiple integral

$$\iint .. dx_1 dx_2 .. dx_n x_1^{\alpha-1} x_2^{\beta-1} .. x_n^{\gamma-1} f(x_1 + x_2 .. + x_n) \ldots (16),$$

the limits of the variables being defined by the inequality

$$x_1 + x_2 .. + x_n \overline{\overline{<}}\, 1.$$

We find $x_1 + x_2 = 1 - u_1 u_2$, $x_1 + x_2 + x_3 = 1 - u_1 u_2 u_3$; and, by induction, $x_1 + x_2 .. + x_n = 1 - u_1 u_2 .. u_n$. This, with the preceding, reduces the integral to the form

$$\iint .. du_1 du_2 .. u_1^{\beta+\gamma ..} u_2^{\gamma+\delta ..} (1 - u_1)^{\alpha-1} (1 - u_2)^{\beta-1} .. f(1 - u_1 u_2 .. u_n) \ldots (17),$$

the limits of each variable being 0 and 1.

Now, in general, $f(t) = t^{\frac{d}{d\theta}} f(\epsilon^\theta)$, if $\theta = 0$. For let $t = \epsilon^\phi$, then $f(t) = f(\epsilon^{\phi+\theta}) = \epsilon^{\phi\frac{d}{d\theta}} f(\epsilon^\theta)$, by Taylor's theorem, $= t^{\frac{d}{d\theta}} f(\epsilon^\theta)$. Hence $f(1 - u_1 u_2 .. u_n) = (u_1 u_2 .. u_n)^{\frac{d}{d\theta}} f(1 - \epsilon^\theta)$. Substituting this expression in (17), we get

$$\iint .. du_1 du_2 \ldots u_1^{\frac{d}{d\theta}+\beta+\gamma\ldots} u_2^{\frac{d}{d\theta}+\gamma+\delta\ldots} \ldots (1-u_1)^{\alpha-1} (1-u_2)^{\beta-1} \ldots f(1-\epsilon^\theta) \ldots (18).$$

But, by the well-known relation connecting the first and the second of the Eulerian integrals,

$$\int_0^1 du_1 u_1^{\frac{d}{d\theta}+\beta+\gamma\ldots} (1 - u_1)^{\alpha-1} = \frac{\Gamma\left(\dfrac{d}{d\theta} + 1 + \beta + \gamma ..\right)\Gamma(\alpha)}{\Gamma\left(\dfrac{d}{d\theta} + 1 + \alpha + \beta ..\right)},$$

$$\int_0^1 du_2 u_2^{\frac{d}{d\theta}+\gamma+\delta\ldots} (1 - u_2)^{\beta-1} = \frac{\Gamma\left(\dfrac{d}{d\theta} + 1 + \gamma ..\right)\Gamma(\beta)}{\Gamma\left(\dfrac{d}{d\theta} + 1 + \beta ..\right)};$$

and finally

$$\int_0^1 du_n u_n^{\frac{d}{d\theta}}(1-u_n)^{\nu-1} = \frac{\Gamma\left(\frac{d}{d\theta}+1\right)\Gamma(\nu)}{\Gamma\left(\frac{d}{d\theta}+1+\nu\right)}.$$

Substituting these values in (18), we have simply

$$\frac{\Gamma(\alpha)\Gamma(\beta)..\Gamma(\nu)\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}+1+\alpha+\beta..\right)} f(1-\epsilon^\theta)$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)..\Gamma(\nu)}{\Gamma(\alpha+\beta..+\nu)}\ \frac{\Gamma(\alpha+\beta..+\nu)\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}+1+\alpha+\beta..\right)} f(1-\epsilon^\theta)$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta)..\Gamma(\nu)}{\Gamma(\alpha+\beta..+\nu} \int_0^1 dv\, v^{\alpha+\beta\ldots+\nu-1}(1-v)^{\frac{d}{d\theta}} f(1-\epsilon^\theta).$$

Now $(1-v)^{\frac{d}{d\theta}} f(1-\epsilon^\theta) = f\{1-(1-v)\} = f(v)$, whence the expression becomes

$$\frac{\Gamma(\alpha)\Gamma(\beta)..\Gamma(\nu)}{\Gamma(\alpha+\beta..+\nu)} \int_0^1 dv\, v^{\alpha+\beta\ldots+\nu-1} f(v).$$

As a final example I propose to consider the definite multiple integral

$$V = \iint .. \frac{dx_1 dx_2 .. dx_{n-1}}{x^n}\{f(a_1x_1 .. + a_{n-1}x_{n-1} + a_nx_n) \\ + f(a_1x_1 .. + a_{n-1}x_{n-1} - a_nx_n)\},$$

the integrations extending to all real values of $x_1 x_2 \ldots x_{n-1}$, and to all real and positive values of x_n which satisfy the conditions

$$x_1^2 + x_2^2 \ldots + x_{n-1}^2 \overline{\overline{<}}\ 1, \qquad x_1^2 + x_2^2 \ldots + x_n^2 = 1.$$

The value of this definite integral is

$$V = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi d\theta \sin\theta^{n-2}\cos\theta\, f(A\cos\theta) \ldots . (20),$$

wherein $A = (a_1^2 + a_2^2 \ldots + a_n^2)^{\frac{1}{2}}$, and it is found by integrating with respect to a_n in the general formula which I have given in a paper entitled, *Remarks on a Theorem of M. Catalan*, (*Journal*, vol. III. p. 281).

Let us assume $x_1 = \cos\theta_1$,

$$x_2 = \sin\theta_1 \cos\theta_2,$$
$$x_{n-1} = \sin\theta_1 \ldots \sin\theta_{n-2}\cos\theta_{n-1},$$
$$x_n = \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-1}.$$

Now while $x_1, x_2 \ldots x_{n-1}$ vary from -1 to 1, the dependent variable x_n is restricted to positive values. These conditions are satisfied by taking 0 and π for the limits of each of the new variables $\theta_1, \theta_2 \ldots \theta_{n-1}$. Considering the $n-1$ first equations of the preceding system, we have $E = 0$, and differentiating with respect to $\theta_1, \theta_2 \ldots \theta_{n-1}$,

$$-\sin\theta_1 d\theta_1 = 0,$$

$$\cos\theta_1 \cos\theta_2 d\theta_1 - \sin\theta_1 \sin\theta_2 d\theta_2 = 0,$$

$$\ldots\ldots - \sin\theta_1 \ldots \sin\theta_{n-1}\, d\theta_{n-1} = 0;$$

whence $E' = \sin\theta_1^{n-1} \sin\theta_2^{n-2} \ldots \sin\theta_{n-1}$, wherefore

$$\frac{dx_1 dx_2 \ldots dx_{n-1}}{x_n} = \frac{\sin\theta_1^{n-1} \ldots \sin\theta_{n-1}}{\sin\theta_1 \ldots \sin\theta_{n-1}} d\theta_1 d\theta_2 \ldots d\theta_{n-1}$$

$$= \sin\theta_1^{n-2} \sin\theta_2^{n-3} \ldots \sin\theta_{n-2}\, d\theta_1 d\theta_2 \ldots d\theta_{n-1},$$

and the transformed integral may be thus written:

$$\int_0^\pi \int_0^\pi \ldots d\theta_1 \ldots d\theta_{n-1} \sin\theta_1^{n-2} \ldots \sin\theta_{n-2} f(a_1 \cos\theta_1 \ldots$$
$$+ a_{n-1} \sin\theta_1 \ldots \sin\theta_{n-2} \cos\theta_{n-1} + a_n \sin\theta_1 \ldots \sin\theta_{n-1})$$
$$+ \int_0^\pi \int_0^\pi \ldots d\theta_1 \ldots d\theta_{n-1} \sin\theta_1^{n-2} \ldots \sin\theta_{n-2} f(a_1 \cos\theta_1 \ldots$$
$$+ a_{n-1} \sin\theta_1 \ldots \sin\theta_{n-2} \cos\theta_{n-1} - a_n \sin\theta_1 \ldots \sin\theta_{n-1}) \ldots (20).$$

In the second term of the above expression let $\theta_{n-1} = 2\pi - \theta'_{n-1}$, then $d\theta_{n-1} = -d\theta'_{n-1}$, $\cos\theta'_{n-1} = \cos\theta'_{n-1}$, $\sin\theta_{n-1} = -\sin\theta'_{n-1}$, also the limits of θ'_{n-1} are 2π and π, wherefore the proposed term becomes

$$-\int_0^\pi \int_0^\pi \ldots \int_{2\pi}^\pi d\theta_1 \ldots d\theta'_{n-1} \sin\theta_1^{n-2} \ldots \sin\theta_{n-2}$$
$$f(a_1 \cos\theta_1 \ldots + a_{n-1} \sin\theta_1 \ldots \cos\theta'_{n-1} + a_n \sin\theta_1 \ldots \sin\theta'_{n-1})$$
$$= \int_0^\pi \int_0^\pi \ldots \int_\pi^{2\pi} d\theta_1 \ldots d\theta'_{n-1} \sin\theta_1^{n-2} \ldots \sin\theta_{n-2}$$
$$f(a_1 \cos\theta_1 \ldots + a_{n-1} \sin\theta_1 \ldots \cos\theta'_{n-1} + a_n \sin\theta_1 \ldots \sin\theta'_{n-1}).$$

Taking away the accent from θ'_{n-1} in the above term, and annexing the result to the first term of (20), we have

$$\int_0^\pi \int_0^\pi \ldots \int_0^{2\pi} d\theta_1 \ldots d\theta_{n-1} \sin\theta_1^{n-2} \ldots \sin\theta_{n-2}$$
$$f(a_1 \cos\theta_1 \ldots + a_{n-1} \sin\theta_1 \ldots \cos\theta_{n-1} + a_n \sin\theta_1 \sin\theta_{n-1})$$
$$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi d\theta \sin\theta^{n-2} \cos\theta\, f(A\cos\theta) \ldots (21),$$

which is the transformation contemplated.

Some remarkable deductions from the general theorem (20) may properly be noticed here. Let $a_n = 0$, then in the result, writing n for $n-1$, and for x_{n+1} its value $\sqrt{(1-x_1^2-x_2^2 \dots x_n^2)}$, we have

$$\iint \dots dx_1 dx_2 \dots dx_n \frac{f(a_1x_1 + a_2x_2 \dots a_nx_n)}{\sqrt{(1-x_1^2 \dots - x_n^2)}}$$

$$= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\pi d\theta \sin\theta^{n-1} \cos\theta f(A\cos\theta) \dots (22),$$

the integrations in the first member extending to all real values $x_1 x_2 \dots x_n$ subject to the condition

$$x_1^2 + x_2^2 \dots x_n^2 \overline{\lessdot} 1 \dots \dots \dots \dots (23).$$

Performing on both sides of the equation the operation $\left(\frac{d}{da_1}\right)^\alpha \left(\frac{d}{da_2}\right)^\beta \dots \left(\frac{d}{da_n}\right)^\nu$, we arrive at a result which may be thus expressed:

$$\iint \dots dx_1 dx_2 \dots dx_n x_1^\alpha x_2^\beta \dots x_n^\nu \frac{F(a_1x_1 + a_2x_2 \dots a_nx_n)}{\sqrt{(1-x_1^2 \dots - x_n^2)}} \dots \dots (24)$$

$$= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{d}{da_1}\right)^\alpha \left(\frac{d}{da_2}\right)^\beta \dots \left(\frac{d}{da_n}\right)^\nu \int_0^\pi d\theta \sin\theta^{n-1} \cos\theta f(A\cos\theta),$$

wherein

$f(t) = \int\int \dots F(t)\, dt^{\alpha+\beta \dots +\nu}$, $A = (a_1^2 + a_2^2 \dots + a_n^2)^{\frac{1}{2}}$, and $x_1, x_2, \dots x_n$ are subject to the condition (23). The theorem is therefore applicable to every form of F which admits of being integrated in succession, $\alpha + \beta \dots + \nu$ times.

Minster Yard, Lincoln, Aug. 30, 1843.

VII.—ON THE MOTION OF A PISTON AND OF THE AIR IN A CYLINDER.

By G. G. Stokes, B.A. Fellow of Pembroke College.

When a piston is in motion in a cylinder which also contains air, if the motion of the piston be not very rapid, so that its velocity is inconsiderable compared with the velocity of propagation of sound, the motions of the air may be divided into two classes, the one consisting of those which depend directly on the motion of the piston, the other, of those which are propagated with the velocity of sound, and depend on the initial state of the air, or on a breach of continuity in the

motion of the piston. If we suppose the initial velocity and condensation of the air in each section of the cylinder to be given, and also the initial velocity of the piston, both kinds of motion will in general take place, and the solution of the problem will be complicated. If, however, we restrict ourselves to motions of the first class, the approximate solution, though rather long, will be simple. In this case we must suppose the initial velocity and condensation of the air not to be given arbitrarily, but to be connected, according to a certain law which is yet to be found, with the motion of the piston. The problem as so simplified may perhaps be of some interest, as affording an example of the application of the partial differential equations of fluid motion, without requiring the employment of that kind of analysis which is necessary in most questions of that sort. It is, moreover, that motion of the air which it is proposed to consider, which principally affects the motion of the piston.

Conceive an air-tight piston to move in a cylinder which is closed at one end, and contains a mass of air between the closed end and the piston. For more simplicity, suppose the rest of the cylinder to contain no air. Let a point in the closed end be taken for origin, and let x be measured along the cylinder. Let x_1 be the abscissa of the piston; a the initial value of x_1; u the velocity parallel to x of any particle of air whose abscissa is x; p the pressure, ρ the density about that particle; Π the initial mean pressure; p_1 the value of p when $x = x_1$; X a function of x, the accelerating force acting on the air; then for the motion of the air we have

$$\left.\begin{aligned} \frac{1}{\rho}\frac{dp}{dx} &= X - \frac{du}{dt} - u\frac{du}{dx}, \\ \frac{d\rho}{dt} + \frac{d\rho u}{dx} &= 0, \\ \text{and} \qquad p &= k\rho, \end{aligned}\right\} \dots\dots\dots (1),$$

neglecting the variation of temperature.

We have also the conditions

$$u = 0 \quad \text{when} \quad x = 0 \dots\dots\dots\dots (2);$$

$$u = \frac{dx_1}{dt} \quad \text{when} \quad x = x_1 \dots\dots\dots\dots (3),$$

for positive values of t, and

$$\int_0^a pdx = \Pi a \quad \text{when} \quad t = 0 \dots\dots\dots (4).$$

Eliminating ρ from equations (1), we have

$$\frac{1}{p}\frac{dp}{dx}=\frac{1}{k}\left(X-\frac{du}{dt}-u\frac{du}{dx}\right)\ldots\ldots\ldots(5);$$

$$\frac{dp}{dt}+\frac{dpu}{dx}=0\ldots\ldots\ldots\ldots\ldots(6).$$

Now, k being very large, for a first approximation let $\frac{1}{k}$ be neglected; then, integrating (5),

$$p=\phi(t).$$

Substituting in (6), and integrating,

$$u=\psi(t)-\frac{\phi'(t)}{\phi(t)}x.$$

The conditions (2) and (3) give

$$\psi(t)=0\,;\quad \frac{\phi'(t)}{\phi(t)}=-\frac{1}{x_1}\frac{dx_1}{dt};$$

whence $$\phi(t)=\frac{C}{x_1}.$$

Substituting in (4) the value of p when $t=0$, we have

$$\int_0^a \frac{C}{a}\,dx=C=\Pi a\,;$$

whence $$p=\Pi\frac{a}{x_1};\quad u=\frac{x}{x_1}\frac{dx_1}{dt}.$$

Let now, for a second approximation,

$$p=\Pi\frac{a}{x_1}+p'\,;\quad u=\frac{x}{x_1}\frac{dx_1}{dt}+u'\,;$$

so that p' and u' are small quantities of the order $\frac{1}{k}$; then, substituting these values in (5) and (6), remembering that the quantities which are not small must destroy each other, and retaining only small quantities of the first order, we have

$$\frac{dp'}{dx}=\frac{\Pi a}{kx_1}\left(X-\frac{x}{x_1}\frac{d^2x_1}{dt^2}\right)\ldots\ldots\ldots\ldots(7);$$

$$\frac{dp'}{dt}+\frac{1}{x_1}\frac{dx_1}{dt}\frac{dp'x}{dx}+\Pi\frac{a}{x_1}\frac{du'}{dx}=0\ldots\ldots(8);$$

and the conditions (2), (3) and (4) give

$u'=0$ when $x=0$, or $x=x_1$, and t is positive. . . .(9);

$$\int_0^a p'dx=0\ \text{ when }\ t=0\ldots\ldots\ldots\ldots(10).$$

Integrating (7), we have

$$p' = \frac{\Pi a}{kx_1}\left(\int_0^x X dx - \frac{x^2}{2x_1}\frac{d^2x_1}{dt^2}\right) + \omega(t) \;\ldots\ldots\; (11).$$

Substituting the values of p' and of its differential coefficients in (8), and integrating, we obtain

$$u' = \frac{x^3}{6kx_1^2}\frac{d}{dt}\left(x_1\frac{d^2x_1}{dt^2}\right) - \frac{1}{kx_1}\frac{dx_1}{dt}\int_0^x Xx dx - \frac{x}{\Pi a}\frac{d}{dt}\{x_1\omega(t)\} + \zeta(t) \;\ldots\ldots(12).$$

The conditions (9) give $\zeta(t) = 0$;

$$\frac{1}{6k}\frac{d}{dt}\left(x_1\frac{d^2x_1}{dt^2}\right) - \frac{1}{kx_1^2}\frac{dx_1}{dt}\int_0^{x'} Xx dx - \frac{1}{\Pi a}\frac{d}{dt}\{x_1\omega(t)\} = 0;$$

and integrating, we get

$$x_1\omega(t) = \frac{\Pi a x_1}{6k}\frac{d^2x_1}{dt^2} - \frac{\Pi a}{k}\int_a^{x_1}\left(\int_0^{x_1} Xx dx\right)\frac{dx_1}{x_1^2} + C \ldots (13).$$

Putting f for the initial value of $\frac{d^2x_1}{dt^2}$ we have, from (10) and (11),

$$\frac{\Pi}{k}\left(\int_0^a dx\int_0^x X dx - \frac{fa^2}{b}\right) + \omega(0)\,a = 0;$$

and substituting the value of $\omega(0)$ given by this equation in (13), after having made $t = 0$, $x_1 = a$, $\frac{d^2x_1}{dt^2} = f$ in the latter, we have

$$C = -\frac{\Pi}{k}\int_0^a dx\int_0^x X dx.$$

Substituting this value of C in that of $\omega(t)$, and substituting in (11) and (12), and then substituting the values of p' and u' in those of p and u, we have

$$p = \Pi\frac{a}{x_1} + \frac{\Pi a}{kx_1}\left(\int_0^x X dx - \frac{x^2}{2x_1}\frac{d^2x_1}{dt^2}\right)$$
$$+ \frac{\Pi a}{6k}\frac{d^2x_1}{dt^2} - \frac{\Pi a}{kx_1}\int_a^{x_1}\left(\int_0^{x_1} Xx dx\right)\frac{dx_1}{x_1^2} - \frac{\Pi}{kx_1}\int_0^a\left(\int_0^x X dx\right)dx \;\ldots.(14);$$

$$u = \frac{x}{x_1}\frac{dx_1}{dt} - \frac{x}{6k}\left(1 - \frac{x^2}{x_1^2}\right)\frac{d}{dt}\left(x_1\frac{d^2x_1}{dt^2}\right)$$
$$+ \frac{1}{kx_1}\frac{dx_1}{dt}\left\{\frac{x}{x_1}\int_0^{x_1} Xx dx - \int_0^x Xx dx\right\} \ldots.(15).$$

Let A be the area of a section of the cylinder, and let $\dfrac{\Pi A a}{k} = \mu$, so that μ is the mass of the air; then we have

$$p_1 A = \Pi A \frac{a}{x_1} - \frac{\mu}{3}\frac{d^2 x_1}{dt^2}$$
$$+ \frac{\mu}{x_1}\int_0^{x_1} X dx - \frac{\mu}{x_1}\int_0^{x_1}\left(\int_0^{x_1} X x dx\right)\frac{dx_1}{x_1^2} - \frac{\mu}{a x_1}\int_0^{a} dx \int_0^{x} X dx.$$

If there were no motion, the term $-\dfrac{\mu}{3}\dfrac{d^2 x_1}{dt^2}$ would disappear. But in that case the value of $p_1 A$, the pressure on the piston, might be deduced immediately from the equation of equilibrium of an elastic fluid

$$\frac{1}{p}\frac{dp}{dx} = \frac{X}{k}.$$

Integrating this equation, determining the constant by the condition that $\int_0^{x_1} p dx = \Pi a$, multiplying by A, and putting $x = x_1$, we have, neglecting $\dfrac{1}{k^2}$,

$$p_1 A = \Pi A \frac{a}{x_1} + \frac{\mu}{x_1}\int_0^{x_1} X dx - \frac{\mu}{x_1^2}\int_0^{x_1}\left(\int_0^{x} X dx\right) dx.$$

Comparing this expression with the above, when the second term of the latter is left out, we have

$$\int_a^{x_1}\left(\int_0^{x_1} X x dx\right)\frac{dx_1}{x_1^2} + \frac{1}{a}\int_0^{a} dx \int_0^{x} X dx = \frac{1}{x_1}\int_0^{a_1} dx \int_0^{a} X dx,$$

a formula which may also be proved directly. We have then

$$p_1 A = \Pi A \frac{x}{x_1} - \frac{\mu}{3}\frac{d^2 x_1}{dt^2} + \mu \frac{d}{dx_1}\left(\frac{1}{x_1}\int_0^{x_1} dx_1 \int_0^{x_1} X dx\right).$$

The first term would be the value of the pressure on the piston if the air had no inertia and were acted on by no external forces; the second term is that due to the *inertia* of the air; the last term is that due to the external forces, and in the case of gravity expresses the effect of the *weight* of the air. If M be the mass of the piston, P the accelerating force parallel to x acting on it, not including the pressure of the air, its equation of motion is

$$\left(M + \frac{\mu}{3}\right)\frac{d^2 x_1}{dt^2} = MP + \Pi A \frac{a}{x_1} + \mu \frac{d}{dx_1}\left(\frac{1}{x_1}\int_0^{x_1} dx_1 \int_0^{x_1} X dx\right) \ldots (16).$$

Hence the effect of the inertia of the air is to increase the

mass of the piston by one third of that of the air, without increasing the moving force acting on it. If we could integrate equation (16) twice, we should determine the arbitrary constants by means of the initial values of x_1 and $\frac{dx_1}{dt}$, and thus get x_1 in terms of t: then, substituting in (14) and (15), we should obtain p and u as functions of x and t.

If the cylinder be vertical and smooth and turned upwards, we have $P = X = -g$; and if, moreover, the motion be very small, putting $x_1 = a + y$, and neglecting y^2, we have

$$\left(M + \frac{\mu}{3}\right) \frac{d^2y}{dt^2} + \frac{\Pi A}{a} y = \Pi A - \left(M + \frac{\mu}{2}\right) g.$$

The term at the second side of this equation is by hypothesis small, and, if we suppose the mean value of x to be taken for a, it is zero. On this supposition $\Pi A = \left(M + \frac{\mu}{2}\right) g$, and the time of a small oscillation will be $2\Pi \sqrt{\frac{M + \frac{\mu}{3}}{M + \frac{\mu}{2}} \cdot \frac{a}{g}}$, which becomes, since μ^2 is neglected throughout, $2\Pi \left(1 - \frac{\mu}{12M}\right)\sqrt{\frac{a}{g}}$.

The reader who wishes to see the complete solution of the problem, in the case where no forces act on the air, and the air and piston are at first at rest, may consult a paper of Lagrange's with additions made by Poisson in the *Journal de l'Ecole Polytechnique.*

VIII.—ON THE EQUATIONS OF THE MOTION OF HEAT REFERRED TO CURVILINEAR CO-ORDINATES.

LET x, y, z be the rectangular co-ordinates of any point in space, and let $\lambda, \lambda_1, \lambda_2$ be any functions of x, y, z, such that

$$\lambda = a, \quad \lambda_1 = a_1, \quad \lambda_2 = a_2. \ldots\ldots\ldots (1)$$

are the equations of three surfaces cutting one another at right angles for any values of the variable parameters a, a_1, a_2. The three series produced by giving all possible values to these parameters form what is called a system of conjugate orthogonal surfaces.

It has been proved by Dupin that the surfaces of any two of the three series cut each surface of the other series in its

lines of curvature. Hence if one series be given, the other two are determinable from it, except in such extreme cases as those in which the lines of curvature of the given series are indeterminate.

In the method of curvilinear co-ordinates, as proposed by Lamé, the position of any point in space is determined by the three conjugate surfaces of the system (1) which intersect in the point. Thus, if any point P be given, there will be at least one of the surfaces of the first series which passes through it, and in general, but especially such cases as we shall consider, there will be only one. a_1 the value of λ corresponding to this surface is one of the co-ordinates of P. The other two co-ordinates are the values of λ_1 and λ_2 corresponding to the surfaces of the second and third series which contain the two lines of curvature through P of the first surface.

This general method of co-ordinates comprehends the two systems, rectangular and polar, in ordinary use. Thus, if $\lambda = x$, $\lambda_1 = y$, $\lambda_2 = z$, equations (1) will become

$$x = a, \quad y = a_1, \quad z = a_2,$$

the equations to three planes at right angles, by their intersection, determining the point P, whose rectangular co-ordinates are a, a_1, a_2. Similarly, if the first be a series of concentric spheres, the second a series of planes through a diameter of the sphere, and the third a series of cones having this diameter for axis, and the centre for vertex, $\lambda, \lambda_1, \lambda_2$ will be polar co-ordinates.

The equations of the motion of heat in a solid body may be referred to the general system of curvilinear co-ordinates in the following manner, which is exactly similar to that by which Fourier establishes them for rectangular rectilinear co-ordinates. For simplicity we shall suppose the body homogeneous, though the investigation would be in principle as simple if this were not the case. Let d be its density, h its conducting power, and c its capacity for heat, or the quantity of heat necessary to raise the temperature of a unit of its mass by unity.

Let $\lambda, \lambda_1, \lambda_2$ be the co-ordinates of any point P in the body, and let $\lambda + d\lambda$, $\lambda_1 + d\lambda_1$, $\lambda_2 + d\lambda_2$ be those of an adjacent point P'. The portions of the six surfaces corresponding to λ, $\lambda + d\lambda$, λ_1, $\lambda_1 + d\lambda_1$, λ_2, and $\lambda_2 + d\lambda_2$ adjacent to the points P and P' will form a rectangular parallelopiped, of which P and P' are opposite angles. Let dp, dp_1, dp_2 be the three edges of this parallelopiped which are respectively

perpendicular to the surfaces corresponding to λ, λ_1, and λ_2: dp will be the element of the normal to the surface λ, commencing at this surface and terminated by the surface $\lambda + d\lambda$, and similarly with dp_1 and dp_2. Hence, by a known theorem,

$$dp = \left(\frac{d\lambda^2}{dx^2} + \frac{d\lambda^2}{dy^2} + \frac{d\lambda^2}{dz^2}\right)^{-\frac{1}{2}} d\lambda = Hd\lambda, \text{ for brevity} \ldots (a),$$

$$dp_1 = \left(\frac{d\lambda_1^2}{dx^2} + \frac{d\lambda_1^2}{dy^2} + \frac{d\lambda_1^2}{dz^2}\right)^{-\frac{1}{2}} d\lambda_1 = H_1 d\lambda_1 \ldots\ldots\ldots\ldots (b),$$

$$dp_2 = \left(\frac{d\lambda_2^2}{dx^2} + \frac{d\lambda_2^2}{dy^2} + \frac{d\lambda_2^2}{dz^2}\right)^{-\frac{1}{2}} d\lambda_2 = H_2 d\lambda_2 \ldots\ldots\ldots\ldots (c).$$

Let v be the temperature of the body at P, and let $v + d_\lambda v + d_{\lambda_1} v + d_{\lambda_2} v$ be the temperature at P'; $d_\lambda v$, $d_{\lambda_1} v$, and $d_{\lambda_2} v$ denoting the increments of v which correspond to the increments $d\lambda$, $d\lambda_1$, $d\lambda_2$ of λ, λ_1, λ_2. This notation, for *partial differentials*, we shall find it convenient to use in those cases in which it is not convenient to employ *partial differential coefficients*. The quantity of heat which flows into the rectangular parallelopiped $dp \,.\, dp_1 \,.\, dp_2$ across the side which coincides with λ, in the time dt, is

$$- k \frac{d_\lambda v}{dp} dp_1 dp_2 \,.\, dt,$$

and the quantity which flows out across the opposite side, in the same time,

$$- k \left(\frac{d_\lambda v}{dp} dp_1 dp_2\right) dt - k d_\lambda \left(\frac{d_\lambda v}{dp} dp_1 dp_2\right) dt.$$

The difference of these two expressions,

$$\text{or } \; k d_\lambda \left(\frac{d_\lambda v}{dp} dp_1 dp_2\right) dt,$$

is the whole quantity of heat which flows into the element during the time dt in a direction perpendicular to λ.

Similarly, the quantities of heat which the element gains in the same time, by motion in directions perpendicular to λ_1 and λ_2, are

$$k d_{\lambda_1} \left(\frac{d_{\lambda_1} v}{dp_1} dp dp_2\right) dt, \quad \text{and } \; k d_{\lambda_2} \left(\frac{d_{\lambda_2} v}{dp_2} dp dp_1\right) dt,$$

and therefore the entire quantity of heat gained by the element in the time dt, is

$$k \left\{ d_\lambda \left(\frac{d_\lambda v}{dp} dp_1 dp_2\right) + d_{\lambda_1} \left(\frac{d_{\lambda_1} v}{dp_1} dp dp_2\right) + d_{\lambda_2} \left(\frac{d_{\lambda_2} v}{dp_2} dp dp_1\right) \right\};$$

or, by (*a*), (*b*), (*c*),

$$k\left\{\frac{d}{d\lambda}\left(\frac{dv}{d\lambda}\frac{H_1H_2}{H}\right)+\frac{d}{d\lambda_1}\left(\frac{dv}{d\lambda_1}\frac{HH_2}{H_1}\right)+\frac{d}{d\lambda_2}\left(\frac{dv}{d\lambda_2}\frac{HH_1}{H_2}\right)\right\}d\lambda d\lambda_1 d\lambda_2,$$

where, in accordance with the ordinary notation of partial differential coefficients, the suffixes are omitted in the partial differentials.

Now if dv be the alteration of the temperature of the element $dpdp_1dp_2$, in the time dt, the corresponding addition of heat is

$$c\,.\,d\,.\,dv\,dp\,dp_1dp_2,\quad\text{or}\quad cd\,.\,dv\,.\,HH_1H_2d\lambda\,d\lambda_1d\lambda_2.$$

Hence we have the equation

$$HH_1H_2\frac{dv}{dt}=\frac{k}{c\,.\,d}\left\{\frac{d}{d\lambda}\left(\frac{dv}{d\lambda}\frac{H_1H_2}{H}\right)+\frac{d}{d\lambda_1}\left(\frac{dv}{d\lambda_1}\frac{HH_2}{H_1}\right)+\frac{d}{d\lambda_2}\left(\frac{dv}{d\lambda_2}\frac{HH_1}{H_2}\right)\right\}\ldots\ldots.(2),$$

which expresses fully, by means of curvilinear co-ordinates, the motion of heat in the interior of homogeneous solid bodies.

If the motion has continued so long as to have become uniform, then $\frac{dv}{dt}=0$, and the equation becomes

$$\frac{d}{d\lambda}\left(\frac{dv}{d\lambda}\frac{H_1H_2}{H}\right)+\frac{d}{d\lambda_1}\left(\frac{dv}{d\lambda_1}\frac{HH_2}{H_1}\right)+\frac{d}{d\lambda_2}\left(\frac{dv}{d\lambda_2}\frac{HH_1}{H_2}\right)=0\ldots(3).$$

This equation was first given by Lamé, in a paper entitled "*Mémoire sur les lois de l'Equilibre du Fluide Ethéré*," in the *Journal de l'Ecole Polytechnique*, (Vol. III. cahier XXIII.), who deduced it by a very laborious transformation from the equation

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}=0\ldots\ldots\ldots\ldots(4),$$

in which the motion is referred to rectilinear co-ordinates. Equation (3) comprehends, as particular forms, equation (4), and the equation in which the motion is referred to polar co-ordinates. The former of these is obtained at once, if we put $\lambda=x$, $\lambda_1=y$, $\lambda_2=z$, which gives $H=H_1=H_2=1$. To find the latter, let

$$\lambda=(x^2+y^2+z^2)^{\frac{1}{2}}=r,$$

$$\lambda_1=\tan^{-1}\frac{y}{x}=\phi,$$

$$\lambda_2=\frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}}=\cos\theta.$$

Hence

$$H = \left(\frac{dr^2}{dx^2} + \frac{dr^2}{dy^2} + \frac{dr^2}{dz^2}\right)^{-\frac{1}{2}} = 1,$$

$$H_1 = \left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right)^{-\frac{1}{2}} = r \sin\theta,$$

$$H_2 = \left\{\frac{(d\cos\theta)^2}{dx^2} + \frac{(d\cos\theta)^2}{dy^2} + \frac{(d\cos\theta^2)}{dz^2}\right\}^{-\frac{1}{2}} = \frac{r^2}{(r^2 - z^2)^{\frac{1}{2}}} = \frac{r}{\sin\theta}.$$

Therefore (3) becomes

$$\frac{d}{dr}\left(r^2 \frac{dv}{dr}\right) + \frac{d}{d\cos\theta}\left(\sin^2\theta \frac{dv}{d\cos\theta}\right) + \frac{d}{d\phi}\left(\frac{1}{\sin^2\theta}\frac{dv}{d\phi}\right) = 0,$$

$$\text{or} \quad r\frac{d^2(rv)}{dr^2} + \frac{d}{d\cos\theta}\left(\sin^2\theta \frac{dv}{d\cos\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2v}{d\phi^2} = 0,$$

the well-known equation of which so much use is made in the determination of the properties of the expansion of v in a series of powers of r. Equation (3) may also be applied to find the conditions which must be satisfied so that each series of a system of conjugate orthogonal surfaces may be isothermal; and we shall thus be enabled to answer the question proposed in a paper in the last number of this *Journal* (vol. III. p. 286).

If the first series of surfaces, or the series represented by the equation $\lambda = a$, be isothermal, (3) must be satisfied by the assumption $v = f(\lambda)$, and therefore

$$\frac{d}{d\lambda}\left(\frac{H_1 H_2}{H} f'\lambda\right) = 0 \ldots\ldots\ldots\ldots (5).$$

Similarly, if the second and third series be isothermal,

$$\frac{d}{d\lambda_1}\left(\frac{H H_2}{H_1} f'_1\lambda_1\right) = 0 \ldots\ldots\ldots\ldots (6),$$

$$\frac{d}{d\lambda_2}\left(\frac{H H_1}{H_2} f'_2\lambda_2\right) = 0 \ldots\ldots\ldots\ldots (7).$$

Integrating these equations, we have

$$\frac{H_1 H_2}{H} f'\lambda = F(\lambda_1, \lambda_2) \ldots\ldots\ldots\ldots (8),$$

$$\frac{H H_2}{H_1} f'_1\lambda_1 = F_1(\lambda, \lambda_2) \ldots\ldots\ldots\ldots (9),$$

$$\frac{H H_1}{H_2} f'_2\lambda_2 = F_2(\lambda, \lambda_1) \ldots\ldots\ldots\ldots (10).$$

These three equations, together with the following,

$$\frac{d\lambda_1}{dx}\frac{d\lambda_2}{dx}+\frac{d\lambda_1}{dy}\frac{d\lambda_2}{dy}+\frac{d\lambda_1}{dz}\frac{d\lambda_2}{dz}=0\ \ldots\ldots\ (11),$$

$$\frac{d\lambda_2}{dx}\frac{d\lambda}{dx}+\frac{d\lambda_2}{dy}\frac{d\lambda}{dy}+\frac{d\lambda_2}{dz}\frac{d\lambda}{dz}=0\ \ldots\ldots\ (12),$$

$$\frac{d\lambda}{dx}\frac{d\lambda_1}{dx}+\frac{d\lambda}{dy}\frac{d\lambda_1}{dy}+\frac{d\lambda}{dz}\frac{d\lambda_1}{dz}=0\ \ldots\ldots\ (13),$$

which make the surfaces cut one another at right angles, are the conditions which must be satisfied if the three series be orthogonal and isothermal. If we could eliminate all the quantities depending on two of the series, the second and third for instance, we should find two equations relative to the first series which make it both be isothermal itself, and possess the property that the two series which cut its surfaces orthogonally shall also be isothermal. This elimination is probably quite impracticable in general. There is however an extensive class of surfaces which we see by inspection satisfies each condition, the class of cylindrical isothermal surfaces. For, if the axis of z be parallel to the generating lines of a series of isothermal cylindrical surfaces represented by the equation $\lambda = a$, λ will be independent of z, and $\lambda_1 = a_1$ being the equation of a series of orthogonal cylindrical surfaces, λ_1 will also be independent of z. The third series of the conjugate system will be a series of planes parallel to xy. Hence H and H_1 are functions of x and y alone, and therefore of λ and λ_1, and H_2 is a function of z and therefore of λ_2. Now since $\lambda = a$ is an isothermal system, (5) must be satisfied, and therefore $\dfrac{H_1H_2}{H}$ must contain the factor $\dfrac{1}{f'(\lambda)}$, and λ must enter in no other manner. Hence $\dfrac{H_1}{H}$ is the product of two functions, one of λ and the other of λ_1, and therefore $\dfrac{HH_2}{H_1}$ must consist of three factors, functions of λ, λ_1, λ_2 separately. Hence if we choose the arbitrary function $f'(\lambda_1)$ properly, (6) will be satisfied. Also, in every case, whether $\lambda = a$ be isothermal or not, we may choose $f'_2(\lambda_2)$ in such a manner that (7) shall be satisfied; in fact a series of parallel planes is necessarily isothermal. We have thus seen that the two conjugate orthogonal series to a series of isothermal cylinders are themselves isothermal, which agrees with what was proved in the paper already alluded to, (vol. III. p. 286).

In the case in which λ and λ_1 are surfaces of revolution we can obtain from (5), (6), and (7) a simple condition, which being satisfied, λ and λ_1 will each be isothermal, if one of them is so. To effect this, let the axis of revolution be taken for axis of x, and let $\rho = (y^2+z^2)^{\frac{1}{2}}$. The generating lines of the surfaces of revolution of which the series λ and λ_1 are composed will be plane curves expressed by two equations between x and ρ. If in these equations we write $(y^2+z^2)^{\frac{1}{2}}$ for ρ, the results will be the equations $\lambda = a$, $\lambda_1 = a_1$ of the surfaces of revolution. Hence λ and λ_1 are functions of x and ρ, and therefore, conversely, x and ρ are functions of λ and λ_1. Hence we see readily that H and H_1 depend only on x and ρ, or on λ and λ_1. Also, the equation $\lambda_2 = a_2$ must represent a series of planes passing through the axis of x, and we may therefore assume $\lambda_2 = \omega = \tan^{-1}\dfrac{y}{z}$. From this we readily deduce $H_2 = \rho$. Hence (7) is always satisfied, independently of λ and λ_1; that is to say, a series of planes passing through a fixed axis, is isothermal. Hence we have only to consider the conditions relative to λ and λ_1, or equations (5) and (6). If we substitute for H_2 its value, these become

$$\frac{d}{d\lambda}\left\{\frac{\rho H_1}{H} f'(\lambda)\right\} = 0 \;\ldots\ldots\ldots\ldots\; (14),$$

$$\frac{d}{d\lambda_1}\left\{\frac{\rho H}{H_1} f'(\lambda_1)\right\} = 0 \;\ldots\ldots\ldots\ldots\; (15).$$

From these we readily deduce

$$\rho = F(\lambda) . F_1(\lambda_1) \;\ldots\ldots\ldots\ldots\; (16),$$

F and F_1 being entirely arbitrary. Hence, if the equation $\lambda = a$ of a series of surfaces of revolution be given and $\lambda_1 = a_1$, the equation to the orthogonal system be deduced, and if it be found that the distance of any point from the axis of revolution can be expressed in the form (13), then both systems, or neither, will be isothermal.

If the given series be isothermal this test may be applied very readily, as in that case we are always able at once to find the equation of the conjugate series. To shew this, let the series λ be isothermal. Then (14) will be true, and therefore we have, by integration,

$$\frac{\rho H_1}{H} f'(\lambda) = F(\lambda_1),$$

$$\therefore \quad H_1 = \frac{HF(\lambda_1)}{\rho f'(\lambda)} \ldots\ldots\ldots\ldots\ldots (17).$$

Also, since the sections of the two surfaces made by any plane through x are perpendicular to one another, we have

$$\frac{d\lambda}{dx}\frac{d\lambda_1}{dx} + \frac{d\lambda}{d\rho}\frac{d\lambda_1}{d\rho} = 0 \ldots\ldots\ldots\ldots (18);$$

this equation gives

$$\frac{\dfrac{d\lambda_1}{dx}}{\dfrac{d\lambda}{d\rho}} = -\frac{\dfrac{d\lambda_1}{d\rho}}{\dfrac{d\lambda}{dx}} = \frac{H}{H_1},$$

and therefore, by (14),

$$F(\lambda_1)\frac{d\lambda_1}{dx} = \frac{dL_1}{dx} = \rho f'(\lambda)\frac{d\lambda}{d\rho},$$

$$F(\lambda_1)\frac{d\lambda_1}{d\rho} = \frac{dL_1}{d\rho} = -\rho f'(\lambda)\frac{d\lambda}{dx},$$

if $L_1 = \int F(\lambda_1)\,d\lambda_1$; therefore

$$L_1 = \int \rho f'(\lambda)\left(\frac{d\lambda}{d\rho}\,dx - \frac{d\lambda}{dx}\,d\rho\right) \ldots\ldots\ldots (19),$$

and $L_1 = a$ is the equation of the series of orthogonal trajectories to the series of curves in which the surfaces of revolution of the given isothermal system are cut by any plane through the axis. We may verify this solution by observing that the criterion of integrability for the expressions given above for $\dfrac{dL_1}{dx}$ and $\dfrac{dL_1}{d\rho}$ is satisfied; since, if we transform the equation

$$\frac{d^2(f\lambda)}{dx^2} + \frac{d^2(f\lambda)}{dy^2} + \frac{d^2(f\lambda)}{dz^2} = 0,$$

to the independent variables x and ρ, $f\lambda$ being independent of $\dfrac{y}{z}$, we have

$$f'(\lambda)\left(\frac{d^2\lambda}{dx^2} + \frac{d^2\lambda}{d\rho^2}\right) + f''(\lambda)\left(\frac{d\lambda^2}{dx^2} + \frac{d\lambda^2}{d\rho^2}\right) + \frac{1}{\rho}f'(\lambda)\frac{d\lambda}{d\rho} = 0.$$

If λ itself satisfy (4), we may take $f(\lambda) = \lambda$, and the transformed equation becomes

$$\frac{d^2\lambda}{dx^2} + \frac{d^2\lambda}{d\rho^2} + \frac{1}{\rho}\frac{d\lambda}{d\rho} = 0 \ldots\ldots\ldots\ldots (20).$$

Also, if we take $\lambda_1 = L_1$, (19) becomes

$$\lambda_1 = \int \rho \left(\frac{d\lambda}{d\rho}\,dx - \frac{d\lambda}{dx}\,d\rho\right) \dots\dots\dots (21).$$

As an example of the application of these formulæ, let λ represent the series of *surfaces of equilibrium* in the case of a sphere having matter distributed over it, according to the law of the distribution of electricity on a neutral conducting sphere under the influence of a distant electrified body. It is readily shown that, if λ be proportional to the *potential* of such a system, on external points, and if the centre of the sphere be origin, and the line joining this point and the influencing body axis of x, we have

$$\lambda = \frac{x}{(x^2 + \rho^2)^{\frac{3}{2}}} = \frac{x}{r^3} \dots\dots\dots\dots (a).$$

Hence $\lambda = a$ represents a series of isothermal surfaces of revolution, and λ satisfies (20), as may be readily verified. Hence, if $\lambda_1 = a_1$ be the orthogonal system of surfaces of revolution, we have, by (21),

$$\lambda_1 = \frac{\rho^2}{(x^2 + \rho^2)^{\frac{3}{2}}} = \frac{\rho^2}{r^3} \dots\dots\dots\dots\dots (b).$$

If between (a) and (b) we eliminate x, we have

$$\rho^{\frac{8}{3}} + \frac{\lambda_1^{\,2}}{\lambda^2}\rho^{\frac{2}{3}} - \frac{\lambda_1^{\,\frac{4}{3}}}{\lambda^2} = 0 \dots\dots\dots\dots (c).$$

The value of ρ deduced from this equation is not of the form $F(\lambda) . F_1(\lambda_1)$; and hence, by (16), the second series is not isothermal. Similarly, if λ represent the potential of two equal material points situated at the distance $2a$ from one another on the point xyz, we have

$$\lambda = \frac{1}{\{(x-a)^2 + \rho^2\}^{\frac{1}{2}}} + \frac{1}{\{(x+a)^2 + \rho^2\}^{\frac{1}{2}}} \dots\dots (d).$$

Then, by (18), we have, for the orthogonal system,

$$\lambda_1 = \frac{x-a}{\{(x-a)^2 + \rho^2\}^{\frac{1}{2}}} + \frac{x+a}{\{(x+a)^2 + \rho^2\}^{\frac{1}{2}}} \dots\dots (e).$$

The value of ρ deduced from these two equations cannot, I think, be of the form $F(\lambda)$, $F(\lambda_1)$, and hence in this case also the second series is not isothermal. Hence the question proposed in the paper already referred to (vol. III. p. 286)

must be answered in the negative, the proposition to which it refers being not generally true, since we have found cases of surfaces of revolution with regard to which it does not hold; but it holds with regard to every system of conjugate orthogonal surfaces, for which equations (8), (9), and (10) are satisfied. Also, since it does not appear that any two of these equations imply the third, it may happen that two series of a system may be isothermal and the third not, as is exemplified in the particular cases above considered, in each of which a series of surfaces of revolution and of orthogonal planes are isothermal, and the third series, consisting of surfaces of revolution, is not isothermal.

P. Q. R.

IX.—OF ASYMPTOTES TO ALGEBRAIC CURVES.

By D. F. Gregory, M.A. Fellow of Trinity College.

The ordinary method of deducing the equations to asymptotes to plane curves, whether by finding finite values of the intercepts of the tangents for infinite values of the co-ordinates of the point of contact, or by the more convenient method of expansion in descending powers of one of the variables, are essentially unsymmetrical. Moreover the former is often inapplicable from the difficulty of finding the true value of a fraction of which the numerator and denominator are infinite, and the latter fails when the asymptote is parallel to one of the co-ordinate axes. The following method, though appropriate to algebraic curves only, is for them of very easy application, and, besides being symmetrical, leads us readily to the demonstrations of various properties of asymptotes to curves.

Let the equation to the curve, cleared of fractions and radicals, be put in the form

$$u = \phi_n(x, y) + \phi_{n-1}(x, y) + \phi_{n-2}(x, y) + \&c. + \phi_0 = 0 \dots (1),$$

the symbol $\phi_r(x, y)$ denoting generally a *homogeneous* function of r dimensions in x and y; then the equation to the tangent at a point (x, y) may be written

$$x' \frac{du}{dx} + y' \frac{du}{dy} + \phi_{n-1}(x, y) + 2\phi_{n-2}(x, y) + \&c. + n\phi_0 = 0 \dots (2),$$

x', y' being the current co-ordinates of the tangent.

The definition of an asymptote is, that it is a line which, passing through a point at a finite distance from the origin, touches the curve at an infinite distance. Now if x, y be the co-ordinates of the point of contact; x', y' those of a point

through which the line passes; l, m the cosines of the angles which the line makes with the axes, we have

$$\frac{x-x'}{l}=\frac{y-y'}{m}=r\ldots\ldots\ldots\ldots\ldots(3),$$

r being the distance between the point x, y and x', y'. Hence

$$x=x'+lr, \quad y=y'+mr,$$

and substituting these values in (1), it becomes

$$\phi_n(l,m)r^n+\left\{\left(x'\frac{d}{dl}+y'\frac{d}{dm}\right)\phi_n(l,m)+\phi_{n-1}(l,m)\right\}r^{n-1}+\&c.=0\ldots(4).$$

But if the point x, y is at an infinite distance, r_1 must be infinite, which involves the condition that the coefficient of the highest power of r shall vanish; consequently we must have

$$\phi_n(l, m)=0\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

as an equation for determining the *direction* of the asymptotes. Again, if we substitute in (2) for x and y their values derived from (3), we have, arranging in powers of r,

$$\left\{\left(x'\frac{d}{dl}+y'\frac{d}{dm}\right)\phi_n(l, m)+\phi_{n-1}(l, m)\right\}r^{n-1}+\&c.=0\ldots(6).$$

Since the asymptote is by definition a particular case of the tangent, this equation also must give an infinite value for r, which involves the condition

$$\left(x'\frac{d}{dl}+y'\frac{d}{dm}\right)\phi_n(l, m)+\phi_{n-1}(l, m)=0\ldots\ldots(7).$$

Now this equation is linear in x' and y', which are the coordinates of *any* point through which the asymptote passes, that is of any point in the line; so that this equation is in fact the equation to the asymptote, if we substitute in it the relations between l and m, which satisfy equation (5). As from the homogeneity of the terms, l and m finally disappear, and as they are involved exactly as x and y are in the equation to the curve, we may express the process for finding the asymptotes to a curve simply as follows. Let the equation to the curve be put in the form

$$u_n+u_{n-1}+u_{n-2}+\&c.+u_0=0\ldots\ldots\ldots(8),$$

u_r being a homogeneous function in x and y of the r^{th} order, then the equation to the asymptotes will be found by eliminating x and y from

$$x'\frac{du_n}{dx}+y'\frac{du_n}{dy}+u_{n-1}=0$$

by means of the relation between x and y given by the equation

$$u_n = 0.$$

Since the equation $u_n = 0$ is of the n^{th} degree in x and y, it appears that a curve of the n^{th} degree can have n asymptotes and no more. If there be any impossible factors in $u_n = 0$, there are no corresponding asymptotes; and as impossible factors enter the equation by pairs, a curve of the n^{th} degree must have n or $n - 2$ or $n - 4$ or &c. asymptotes.

As an example take the curve

$$y^3 - x^3 - ax^2 = 0.$$

In this case $u_n = 0$ becomes $y^3 - x^3 = 0$, in which there is only one possible factor $y - x = 0$. The other equation

$$3y'y^2 - 3x'x^2 - ax^2 = 0$$

becomes, on putting $y = x$,

$$y' - x' = \frac{a}{3},$$

which is the equation to the asymptote.

Again, let the curve be

$$xy^2 - x^3 + 2a^2y = 0;$$

then the equation $u_n = 0$ becomes

$$xy^2 - x^3 = 0,$$

giving three factors

$$x = 0, \quad y = + x, \quad y = - x.$$

The first of these substituted in the equation

$$x'(y^2 - 3x^2) + 2y'xy = 0,$$

gives

$$x' = 0,$$

the second and third give

$$x' - y' = 0 \quad \text{and} \quad x' + y' = 0,$$

which are the equations to the three asymptotes, the first being evidently the axis of y, and the others inclined at angles $\pm 45^\circ$ to the axis of x. Let the curve be

$$(x + a)y^2 = (y + b)x^2, \quad \text{or} \quad xy^2 - yx^2 + ay^2 - bx^2 = 0.$$

The equation $u_n = 0$ here becomes

$$xy^2 - yx^2 = 0, \quad \text{or} \quad xy(y - x) = 0,$$

which has three possible factors

$$x = 0, \quad y = 0, \quad y - x = 0.$$

The general equation to the asymptotes is

$$x'(y^2 - 2xy) + y'(2xy - x^2) + ay^2 - bx^2 = 0:$$

for $x = 0$ this is reduced to

$$x' + a = 0;$$

for $y = 0$ it becomes $\qquad y' + b;$

and for $y - x = 0$ it becomes

$$y' - x' + a - b = 0,$$

and these are the equations to the three asymptotes. One advantage of this method of finding asymptotes is that a simple inspection of the highest terms of the equation shows at once the number and direction if the asymptotes. The method however fails if the equation $u_n = 0$ contain equal possible roots, indicating the existence of parallel asymptotes. For in this case $\frac{du_n}{dx}$ and $\frac{du_n}{dy}$ will vanish along with u_n, and consequently the equation to the asymptote is nugatory; but a slight extension of the process enables us to overcome the difficulty. It will be seen that the two equations are the coefficients of the n^{th} and $(n-1)^{\text{th}}$ power of r in the expansion (4). In the case of failure from the existence of m equal roots in the equation $u_n = 0$, all that is necessary is to equate to zero the coefficient of the highest power of r which is unaffected by the equality of roots that is the coefficient of the $(n-m)^{\text{th}}$ power of r, and this equation combined with the m equal roots of $u_n = 0$ gives the equations to the parallel asymptotes. If there be two equal roots, and the equation to the curve be put in the form (8), the equation for determining the parallel asymptotes, or the coefficient of r^{n-2} equated to zero, is

$$\frac{1}{2}\left(x'^2\frac{d^2u_n}{dx^2} + 2x'y'\frac{d^2u_n}{dxdy} + y'^2\frac{d^2u_n}{dy^2}\right) + x'\frac{du_{n-1}}{dx} + y'\frac{du_{n-1}}{dy} + u_{n-2} = 0 \ldots (9).$$

As an example take the curve

$$x^2y - x^3 - 3bxy + 2b^2y = 0.$$

Here $u_n = 0$ becomes $x^2y - x^3 = 0$ gives

$$x^2 = 0, \qquad y - x = 0.$$

$$\frac{d^2u_n}{dx^2} = 2y - 6x = 2y \text{ when } x = 0,$$

$$\frac{d^2u_n}{dxdy} = 2x = 0 \text{ when } x = 0, \qquad \frac{d^2u_n}{dy^2} = 0;$$

$$\frac{du_{n-1}}{dx} = -3by, \qquad \frac{du_{n-1}}{dy} = -3bx = 0 \text{ when } x = 0;$$

consequently equation (9) becomes, on dividing by y,

$$x'^2 - 3bx' + 2b^2 = 0,$$

which is decomposable into the two factors

$$x' - b = 0, \qquad x' - 2b = 0,$$

the equations to two asymptotes parallel to the axis of y. The equation to the other asymptote is given by the equation $y - x = 0$ is

$$y' - x' - 3b = 0.$$

In like manner we may shew that the curve

$$x^2 (x^2 + y^2) = a^2 (y - x)^2$$

has two parallel asymptotes, of which the equations are

$$x' + a = 0, \text{ and } x' - a = 0.$$

The equation (4) for any values of l and m is an equation for determining the value of r, the portion of the line whose direction-cosines are l and m intercepted between a given point and the curve. It is generally of n dimensions, so that the line generally meets the curve in n points; but when the line is an asymptote, the first two terms disappear and the equation is reduced to $n - 2$ dimensions. Consequently an asymptote cannot meet its curve in more than $n - 2$ points; and as for all lines parallel to an asymptote the first term of (4) vanishes, lines parallel to an asymptote cannot meet the curve in more than $n - 1$ points.

Since, from what precedes, it appears that the equation to an asymptote depends only on the terms involving the highest and second highest powers of the variables, all curves for which these are the same have the same asymptotes, and *vice versâ*. And as among the curves of the n^{th} order is to be included that made up of the n asymptotes themselves, the product of their n linear equations must have the same highest and second highest terms as the equation to the curve; that is, the equation to the curve differs from the product of the equations to the n asymptotes only in terms of the $(n - 2)^{\text{th}}$ order: so that if the equations to the asymptotes be the n linear equations

$$u' = 0, \quad u'' = 0 \;\ldots\ldots\; u^{(n)} = 0,$$

that to the curves to which these are asymptotes may be written $\quad u'u'' \ldots . u^{(n)} + u_{n-2} + u_{n-3} + \ldots . + u_0 = 0.$

Thus if the curve be of the third order, its equation is

$$u'u''u''' + u_1 + u_0 = 0.$$

When an asymptote meets the curve, which it can do in one point only, this is to be combined with $u' = 0$, $u'' = 0$, $u''' = 0$, any one of which reduces the preceding equation to

$$u_1 + u_0 = 0,$$

which being a linear equation common to the three points in which the curve meets its asymptotes, shews that they lie in one straight line.

In like manner we may shew that the six points in which the curve is cut by lines drawn parallel to the asymptotes all lie in a curve of the second order.

X.—MATHEMATICAL NOTES.

1. If a plane passes through any point of a surface, and makes any function of the intercepts it cuts off from the axes, a maximum or a minimum when it touches the surface, this maximum or minimum value is constant for all points of the surface; and, conversely, if for every point of a surface, a given function of the intercepts of the tangent plane is constant, this function is, with reference to any single point of the surface, a maximum or minimum for the tangent plane.

This appears at once from the following considerations: x, y, z being a point in the surface, x_0, y_0, z_0, the three intercepts, $\phi\,(x_0, y_0, z_0)$ the given function, if we seek to determine the surface so that ϕ shall be a maximum or minimum, we have the equations

$$\frac{x}{x_0}+\frac{y}{y_0}+\frac{z}{z_0}=1 \;\ldots\ldots\ldots\ldots\; (1),$$

$$\frac{d\phi}{dx_0}=\mu\,\frac{x}{x_0^2}\;\ldots\;(2), \qquad \frac{d\phi}{dy_0}=\mu\,\frac{y}{y_0^2}\;\ldots\;(3);$$

$$\frac{d\phi}{dz_0}=\mu\,\frac{z}{z_0^2}\;\ldots\ldots\ldots\;(4),$$

μ being a factor. From these equations we get

$$x_0=f_1(xyz), \qquad y_0=f_2(xyz), \qquad z_0=f_3(xyz),$$

and therefore the differential equation of the surface is

$$\frac{dx}{f_1(xyz)}+\frac{dy}{f_2(xyz)}+\frac{dz}{f_3(xyz)}=0\;\ldots\ldots\ldots\;(5);$$

for by the ordinary equation of the tangent plane we have

$$\frac{1}{x_0}:\frac{1}{y_0}:\frac{1}{z_0}::\frac{dF}{dx}:\frac{dF}{dy}:\frac{dF}{dz},$$

$F=0$ being the equation to the surface.

Again, if we seek to determine the surface, so that ϕ shall be constant, *i.e.* to find the envelope of all the planes represented by (1), we have (1), (2), (3), (4), as before, and in addition

$$\phi\,(x_0, y_0, z_0)=c\;\ldots\ldots\ldots\;(6).$$

Thus, as before, $x_0 = f_1$, $y_0 = f_2$, $z_0 = f_3$, and the equation of the surface may be got by integrating (5), and determining the constant so that the result may coincide with (6). And the identity of the equations connecting x_0, y_0, z_0, and x, y, z, in the two cases proves our proposition and its converse. Take as an example the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad \text{Here } x_0 = \frac{a^2}{x}, \quad y_0 = \frac{b^2}{y}, \quad z_0 = \frac{c^2}{z}.$$

$$\therefore \frac{a^2}{x_0^2} + \frac{b^2}{y_0^2} + \frac{c^2}{z_0^2} = 1.$$

Therefore the tangent plane to any point of the ellipsoid makes $\frac{a^2}{x_0^2} + \frac{b^2}{y_0^2} + \frac{c^2}{z_0^2}$, a minimum with reference to any plane passing through that point.

(ϵ).

2. To find the value of

$$\frac{fa_1}{(a_2 - a_1)(a_3 - a_1)\ldots(a_n - a_1)} + \frac{fa_2}{(a_1 - a_2)(a_3 - a_1)\ldots(a_n - a_2)} + \&\text{c.} = A,$$

when $a_1 = a_2 = \&\text{c.} = a$.

Let $a_1 = a + z_1$, $a_2 = a + z_2$, &c.

$\therefore (a_2 - a_1)(a_3 - a_1)\ldots.(a_n - a_1) = (z_2 - z_1)(z_3 - z_1)\ldots.(z_n - z_1)$, and so of the rest;

$$\therefore A = fa\left\{\frac{1}{(z_2 - z_1)\ldots.(z_n - z_1)} + \frac{1}{(z_1 - z_2)\ldots.(z_n - z_2)} + \&\text{c.}\right\}$$

$$+ \&\text{c.}$$

$$\frac{+ f^{(p)}a}{1.2\ldots p}\left\{\frac{z_1^p}{(z_2 - z_1)\ldots.(z_n - z_1)} + \frac{z_2^p}{(z_1 - z_2)\ldots.(z_n - z_2)} + \&\text{c.}\right\}$$

+ &c. by Taylor's theorem.

Now, when $z_1 = z_2 = \&\text{c.} = 0$, the coefficient of $f^{(p)}a$ will vanish if $p > n$. And whatever the values of z, the coefficient of $f^{(p)}a$ vanishes if $p < n$: for we know that

$$\Sigma \frac{z_1^k}{(z_2 - z_1)\ldots.(z_n - z_1)} = 0,$$

k being $< n$, and $= 1$ when $k = n$;

$$\therefore A = \frac{f^{(n)}a}{1.2\ldots.n},$$

which was to be found.

(ϵ).

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I.—ON THE LUNAR THEORY.

Continued from vol. III. p. 257.

15. WE shall now apply the equations of Article (12) to the Lunar Theory, in the following manner. We shall first suppose that the moon moves in the plane of the ecliptic, and afterwards prove the results obtained on this hypothesis to be true when the inclination is taken into account; we shall then determine the motion of the plane of the orbit, the mean motions of the perigee and node to the third order, and the parallactic inequality.

By Article (12), we have

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{\mu}{r^2} + \frac{dR}{dr} \dots\dots\dots\dots (1)$$

and

$$\frac{dh}{dt} = \frac{dR}{d\theta} \dots\dots\dots\dots\dots\dots (2);$$

and by Article (3), we have

$$\frac{d^2r}{dt^2} = -\frac{d^2u}{d\theta^2}h^2u^2 - \frac{du}{d\theta}\frac{dh}{dt}.$$

Therefore (1) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} - \frac{r^2}{h^2}\frac{dR}{dr} + \frac{1}{h^2}\frac{dr}{d\theta}\frac{dR}{d\theta} \dots\dots (3);$$

and since $\dfrac{d\theta}{dt} = \dfrac{h}{r^2}$ (2) becomes

$$\frac{d(h^2)}{d\theta} = 2r^2\frac{dR}{d\theta} \dots\dots\dots\dots\dots (4).$$

16. To determine R, we have, by Article (14), putting $\cos(r, r') = p$,

$$R = m'\left(\frac{1}{r_1} - \frac{r}{r'^2}p\right),$$

and $\frac{1}{r_1} = \frac{1}{r'}(1 - 2pz + z^2)^{-\frac{1}{2}}$ where $z = \frac{r}{r'}$,

$$= \frac{1}{r'}\left\{1 + \tfrac{1}{2}(2pz - z^2) + \frac{1.3}{2.4}(2pz - z^2)^2 + \frac{1.3.5}{2.4.6}(2pz - z^2)^3 \&\text{c.}\ldots\right\}$$

$$= \frac{1}{r'}\left\{1 + pz + \frac{3p^2 - 1}{2}z^2 + \frac{5p^3 - 3p}{2}z^3 \&\text{c.}\ldots\right\}.$$

Hence, retaining only the lowest power of z or $\frac{r}{r'}$, we have (observing that p or $\cos(r, r') = \cos(\theta - \theta')$, since we suppose the moon to move in the plane of the ecliptic)

$$R = \frac{m'}{r'} + \frac{m'r^2}{r'^3}\,\frac{1 + 3\cos 2(\theta - \theta')}{4},$$

and therefore $\frac{dR}{dr} = \frac{m'r}{r'^3}\cdot\frac{1 + 3\cos 2(\theta - \theta')}{2}$ ...... (5).

$$\frac{dR}{d\theta} = -\frac{m'r^2}{r'^3}\cdot\frac{3\sin 2(\theta - \theta')}{2} \quad \ldots\ldots\ldots (6).$$

We shall expand the second members of (3) and (4) as far as the third order of small quantities, retaining, however, only those terms of the third order in which the coefficient of θ is nearly equal to unity or zero; and this we shall do by substituting for r, r', and θ' their values in terms of θ to the first order, namely

$$r = a\{1 - e\cos(\theta - \omega)\} \quad r' = a'\{1 - e'\cos(\theta' - \omega')\} \ldots (7),$$

$$\theta' = m\theta + \beta + 2e'\sin(m\theta + \beta - \omega') \ \ldots\ldots\ (8).$$

17. To determine $\frac{\mu}{h^2}$ in this manner, we have, by (4) and (6),

$$\frac{d(h^2)}{d\theta} = -\frac{3m'r^4}{r'^3}\sin 2(\theta - \theta');$$

or, by (7), putting $\frac{m'a^3}{\mu a'^3} = m^2$,

$$\frac{d(h^2)}{d\theta} = -3a\mu m^2\{1 - 4e\cos(\theta - \omega) + 3e'\cos(\theta' - \omega')\}\sin 2(\theta - \theta').$$

In the terms of this, which are multiplied by e and e', we may put for θ' its first approximate value, namely, $m\theta + \beta$. In the remaining term, if we put for θ' its value (8), expand to the first power of e', and then transform products of sines and cosines into sums, in the usual way, it is evident that $\sin 2(\theta - \theta')$ will contain no term in which the coefficient of θ is nearly equal to unity or zero. It follows therefore that

we may suppose θ' to be equal to $m\theta + \beta$, simply, in all the terms of this equation.

This being the case, the product $\cos(\theta' - \omega') \sin 2(\theta - \theta')$, transformed into a sum, gives no term to be retained, and the product $\cos(\theta - \omega) \sin 2(\theta - \theta')$ gives only $\frac{1}{2}\sin(\theta - 2\theta' + \omega)$: we have therefore

$$\frac{d(h^2)}{d\theta} = -3a\mu m^2 \{\sin 2(\theta - \theta') - 2e \sin(\theta - 2\theta' + \omega)\}.$$

Therefore, integrating, supplying the proper constant $\{a\mu(1 - e^2)\}$, and neglecting powers and products of m and e above the second order, we have

$$h^2 = a\mu \left\{1 - e^2 + \frac{3m^2}{2}\cos 2(\theta - \theta') - 6m^2 e \cos(\theta - 2\theta' + \omega)\right\},$$

$$\therefore \frac{\mu}{h^2} = \frac{1}{a}\left\{1 + e^2 - \frac{3m^2}{2}\cos 2(\theta - \theta') + 6m^2 e \cos(\theta - 2\theta' + \omega)\right\}.$$

18. To find $-\frac{r^2}{h^2}\frac{dR}{dr}$, we have, by (5) and (7),

$$-\frac{r^2}{h^2}\frac{dR}{dr} = -\frac{m'r^3}{a\mu r'^3}\frac{1 + 3\cos 2(\theta - \theta')}{2}$$

$$= -\frac{m^2}{a}\{1 - 3e\cos(\theta - \omega) + 3e'\cos(\theta' - \omega')\}.\frac{1 + 3\cos 2(\theta - \theta')}{2}.$$

For exactly the same reasons as before, we may here suppose θ' to be equal to $m\theta + \beta$ simply; and then, retaining only the proper terms, we have

$$-\frac{r^2}{h^2}\frac{dR}{dr} = \frac{m^2}{2a}\{-\tfrac{1}{2} - \tfrac{3}{2}\cos 2(\theta - \theta') + \tfrac{3}{2}e\cos(\theta - \omega)$$
$$+ \tfrac{9}{4}e\cos(\theta - 2\theta' + \omega) - \tfrac{3}{2}e'\cos(\theta' - \omega')\}.$$

19. Lastly, to find $\frac{1}{h^2}\frac{dr}{d\theta}\frac{dR}{d\theta}$, we have, by (6) and (7), proceeding exactly as before,

$$\frac{1}{h^2}\frac{dr}{d\theta}\frac{dR}{d\theta} = -\frac{3m'r^2}{a\mu r'^3}\frac{dr}{d\theta}\sin 2(\theta - \theta')$$

$$= -\frac{3m^2}{2a}e\sin(\theta - \omega)\sin 2(\theta - \theta')$$

$$= -\frac{3m^2}{4a}e\cos(\theta - 2\theta' + \omega).$$

20. Let us now, for brevity, put

$$\left.\begin{array}{l} 2(\theta - \theta') = \phi, \quad \theta - 2\theta' + \omega = \psi, \quad \theta' - \omega' = \chi \\ \theta' \text{ here being simply } m\theta + \beta; \end{array}\right\} \dots. (9).$$

Also, for $e\cos(\theta-\omega)$ put $au-1$; and then we have

$$\frac{\mu}{h^2}=\frac{1+e^2}{a}+\frac{m^2}{a}\left\{-\frac{3}{2}\cos\phi+6e\cos\psi\right\}$$

$$-\frac{r^2}{h^2}\frac{dR}{d\theta}=\frac{3m^2}{2}u+\frac{m^2}{a}\left\{-2-\frac{3}{2}\cos\phi+\frac{9}{4}e\cos\psi-\frac{3}{2}e'\cos\chi\right\}$$

$$\frac{1}{h^2}\frac{dr}{d\theta}\frac{dR}{d\theta}=\frac{m^2}{a}\left\{-\frac{3}{4}e\cos\psi\right\}.$$

Hence, (3) becomes

$$\frac{d^2u}{d\theta^2}+\left(1-\frac{3m^2}{2}\right)u$$

$$=\frac{1}{a}\left(1+e^2-2m^2-3m^2\cos\phi+\tfrac{15}{2}m^2e\cos\psi-\tfrac{3}{2}m^2e'\cos\chi.\right)$$

Assume now $\left(\text{putting } 1-\dfrac{3m^2}{2}=c^2\right)$

$$u=\frac{1}{a}\left\{e\cos(c\theta-\omega)+A+B\cos\phi+C\cos\psi+D\cos\omega\right\};$$

and therefore, by (9),

$$\frac{d^2u}{d\theta^2}+c^2u=\frac{1}{a}\left[c^2A+\left\{c^2-(2-2m)^2\right\}B\cos\phi\right.$$
$$\left.+\left\{c^2-(1-2m)^2\right\}C\cos\psi+(c^2-m^2)D\cos\chi\right].$$

Hence, equating coefficients, neglecting powers and products of m and e above the second order, and putting $\dfrac{1}{a}(1+e^2-2m^2)=u_1$, we have

$$u=u_1\left\{1+e\cos(c\theta-\omega)+m^2\cos\phi+\frac{15me}{8}\cos\psi-\frac{3m^2e'}{2}\cos\chi\right\}.$$

21. To find t in terms of θ, we have

$$\frac{dt}{d\theta}=\frac{r^2}{h}\quad\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

in which we must substitute the values of h and r already found in Arts. (17) and (20), and then integrate. But since h is found by an integration with respect to θ, and t by another integration with respect to θ, it is *not* sufficient to retain merely the terms of the third order in which the coefficient of θ is small in the expression for $\dfrac{d(h^2)}{d\theta}$, but we must also preserve all similar terms of the *fourth* order.*

* In Airy's Tracts and Pratt's Mechanical Philosophy the terms of the third order *only* are retained, and therefore it appears to me that t is found by an inconclusive method, so far as terms of the second order are concerned.

To obtain these terms in the expression for $\frac{d(h^2)}{d\theta}$, we must substitute for r^4, r'^3 and θ' their values to the *second* order, namely

$$r^4 = a^4\left\{1+e^2-4e\cos(\theta-\omega)+5e^2\cos 2(\theta-\omega)-4m^2\cos\phi-\frac{15me}{2}\cos\psi\right\},$$

$$r'^3 = a'\{1-3e'\cos(\theta'-\omega)\};$$

(we may look upon e'^2 as of the same order as m^3).

$$\theta' = m\theta+\beta+2e'\sin(m\theta+\beta-\omega')-2me\sin(\theta-\omega).$$

Now it is easy to see that, on account of the expression for $\frac{d(h^2)}{d\theta}$ being multiplied by $\sin 2(\theta-\theta')$, the expansions of r' and θ' will give rise to no terms in which the coefficient of θ is small; we may therefore suppose $r'=a'$ and $\theta'=m\theta+\beta$ simply. But in the expansion of r^4, the term $5e^2\cos 2(\theta-\omega)$ combined with $\sin 2(\theta-\theta')$ gives a term to be retained, namely $-\frac{5}{2}e^2\sin 2(\theta'-\omega)$. We have therefore

$$\frac{d(h^2)}{d\theta} = -3a\mu m^2\{\ldots\ \ldots\ldots -\tfrac{5}{2}e^2\sin 2(\theta'-\omega)\},$$

$$\therefore\quad h^2 = a\mu\{\ldots\ldots -\tfrac{15}{4}me^2\cos 2(\theta'-\omega)\}$$

$$\frac{\mu}{h^2} = \frac{1}{a}\{\ldots\ldots +\tfrac{15}{4}me^2\cos 2(\theta'-\omega)\}.$$

Introducing this additional term in Art. (20), we find

$$u = u_1\{\ldots\ldots\ \tfrac{15}{4}me^2\cos 2(\theta'-\omega)\}.$$

22. We may now proceed to find t.

Represent the values of u and h^2 by $u_1(1+U)$ and $a\mu(1-e^2)(1+H)$; then we have, by (10),

$$\frac{dt}{d\theta} = u^{-2}(h^2)^{-\frac{1}{2}} = \frac{1}{u_1^2\sqrt{\{a\mu(1-e^2)\}}}\cdot(1-2U+3U^2-\tfrac{1}{2}H)\ldots(11).$$

The values of U and H (neglecting in H the term of the third order in which the coefficient of θ is not small) are

$$U = e\cos(c\theta-\omega)+m^2\cos\phi$$
$$+\frac{15me}{8}\cos\psi-\frac{3m^2e'}{2}\cos\chi+\frac{15me^2}{4}\cos 2(\theta'-\omega),$$

$$H = \frac{3m^2}{2}\cos\phi-\frac{15me^2}{4}\cos 2(\theta'-\theta);$$

U^2 evidently contains one term of the second order and two of the third, namely,

$$e^2 \cos^2(c\theta - \omega), \quad 2m^2 e \cos(c\theta - \omega)\cos\phi,$$

$$\text{and} \quad \frac{15me^2}{4}\cos(c\theta - \omega)\cos\psi.$$

The second of these gives no term to be retained, the third gives (putting $c = 1$ and observing that $\psi = \theta - 2\theta' + \omega$), $\frac{15me^2}{8}\cos 2(\theta' - \omega)$, and the first is equal to

$$\tfrac{1}{2}e^2\left\{1 + \cos 2(c\theta - \omega)\right\}.$$

Hence the coefficient of the term $15m^2 e \cos 2(\theta' - \theta)$ in (11) is $-\frac{2}{4} + \frac{3}{8} + \frac{1}{8}$, which $= 0$; this term therefore disappears. Therefore, collecting the terms of (11), and putting all the constant part $= \frac{1}{n}$, we have

$$\frac{ndt}{d\theta} = 1 - 2e\cos(c\theta - \omega)$$
$$+ \frac{3e^2}{2}\cos 2(c\theta - \omega) - \frac{11m^2}{4}\cos\phi - \frac{15me}{4}\cos\psi + 3m^2 e'\cos\chi;$$

and therefore, integrating,

$$nt + \text{const.} = \theta - 2e\sin(c\theta - \omega)$$
$$+ \frac{3e^2}{4}\sin 2(c\theta - \omega) - \frac{11m^2}{8}\sin\phi - \frac{15me}{4}\sin\psi + 3me'\sin\omega.$$

23. We now proceed to shew that these results are not affected by the inclination of the moon's orbit to the plane of the ecliptic.

By Art. (14) p is the only part of R which contains ι; therefore, since, by Art. (16),

$$R = \frac{m'}{r'} + \frac{m'r^2}{r'^3}\;\frac{3p^2 - 1}{2},$$

we have $$\frac{dR}{d\iota} = \frac{3m'r^2}{r'^3}\,p\,\frac{dp}{d\iota} \ldots\ldots\ldots\ldots\ldots (12),$$

and, by Article (14), observing that $\lambda' = 0$, $\theta'_1 = \theta'$,

$$p = \cos\lambda\cos(\theta_1 - \theta'),$$

$$\therefore \frac{dp}{d\iota} = -\sin\lambda\frac{d\lambda}{d\iota}\cos(\theta_1 - \theta') - \cos\lambda\frac{d\theta_1}{d\iota}\sin(\theta_1 - \theta') \ldots (13).$$

Now, by Article (14),

$$\sin\lambda = \sin\iota\sin(\theta - \nu) \qquad \tan(\theta_1 - \nu) = \cos\iota\tan(\theta - \nu),$$

hence, differentiating and then neglecting higher powers of λ and ι, and putting $\theta_1 = \theta$, we find

$$\sin\lambda\,\frac{d\lambda}{d\iota} = \iota\sin^2(\theta-\nu),\quad \cos\lambda\,\frac{d\theta_1}{d\iota} = -\iota\sin(\theta-\nu)\cos(\theta-\nu),$$

therefore, by (13) and (12),

$$\frac{dp}{d\iota} = -\iota\sin(\theta-\nu)\{\sin(\theta-\nu)\cos(\theta-\theta') - \cos(\theta-\nu)\sin(\theta-\theta')\},$$

$$p\,\frac{dp}{d\iota} = -\iota\sin(\theta-\nu)\sin(\theta'-\nu)\cos(\theta-\theta'),$$

$$\frac{dR}{d\iota} = -\frac{3m'r^2}{r'^3}\,\iota\cos(\theta-\theta')\sin(\theta-\nu)\sin(\theta'-\nu) \ldots\ldots (14).$$

Hence, if we denote by R_1 the part of R independent of the inclination, we find, integrating,

$$R = R_1 - \frac{3m'r^2}{2r'^3}\,\iota^2\cos(\theta-\theta')\sin(\theta-\nu)\sin(\theta'-\nu);$$

from which it appears that the part of R depending on the inclination is of the *fourth* order; therefore the only part of the preceding investigation that can be affected by the inclination is the determination of h^2 from $\dfrac{d(h^2)}{d\theta}$, Article (17), and we need only look out for those terms in which the coefficient of θ is small in $\dfrac{dR}{d\theta}$. Now

$$\frac{dR}{d\theta} = \frac{dR_1}{d\theta} - \frac{3m'r^2}{2r'^3}\,\iota^2\{\cos(\theta-\theta')\cos(\theta-\nu) - \sin(\theta-\theta')\sin(\theta-\nu)\}\cos(\theta'-\nu)$$

$$= \frac{dR_1}{d\theta} - \frac{3m'r^2}{2r'^3}\,\iota^2.\cos(2\theta-\theta'-\nu)\sin(\theta'-\nu).$$

The part multiplied by ι^2 here evidently gives no term in which the coefficient of θ is small, we have therefore

$$\frac{dR}{d\theta} = \frac{dR_1}{d\theta};$$

and therefore the value of h^2, found in Article (17), is the correct value when the inclination is taken into account.

Again, in finding the time, instead of $h = r^2\dfrac{d\theta}{dt}$, we have, by Article (12),

$$h = r^2\left(\frac{d\theta}{dt} - 2\sin^2\frac{\iota}{2}\,\frac{d\nu}{dt}\right)$$

$$= r^2\left(\frac{d\theta}{dt} - \frac{\iota}{2h}\,\frac{dR}{d\iota}\right) \text{ by Article (12).}$$

The part involving ι here is of the fourth order, and is therefore to be neglected altogether.

Hence it appears that the supposition that the inclination is zero does not in any way affect the process of finding the values of u and t in terms of θ to the second order of small quantities. It is important to remark that θ is the longitude of the moon measured on the plane of her orbit; and to introduce the longitude measured on the plane of the ecliptic, namely θ_1, we must apply the common *reduction*

$$\theta_1 - \theta = -\frac{\iota^2}{4} \sin 2(\theta - \nu).$$

This explains the reason why the only term in the expression for the time in Airy's Tracts, which depends on the inclination, is the *reduction* just put down.

24. We now proceed to determine the motion of the plane of the moon's orbit.

By Article (12)* we have

$$\frac{d\nu}{d\theta} = \frac{d\nu}{dt}\frac{r^2}{h} = \frac{r^2}{h^2\iota}\frac{dR}{d\iota} \quad \text{and} \quad \frac{d\iota}{dt} = \iota \cot(\theta - \nu)\frac{d\nu}{dt};$$

therefore, by Article (23), equation (12), we have

$$\frac{d\nu}{d\theta} = -3m^2 \cos(\theta - \theta')\sin(\theta - \nu)\sin(\theta' - \nu),$$

$$\frac{d\iota}{d\theta} = -3m^2\iota \sin(\theta - \theta')\sin(\theta - \nu)\sin(\theta' - \nu).$$

In both these put

$$2\sin(\theta - \nu)\sin(\theta' - \nu) = \cos(\theta - \theta') - \cos(\theta + \theta' - 2\nu),$$

retain only the terms which rise in integration, and the constant term in $\frac{d\nu}{d\theta}$, and we find

$$\frac{d\nu}{d\theta} = -\frac{3m^2}{4}\{1 - \cos^2(\theta' - \nu)\},$$

$$\frac{d\iota}{d\theta} = -\frac{3m^2}{4}\iota \sin 2(\theta' - \nu),$$

which give $\nu = \text{const.} - \frac{3m^2}{4}\theta + \frac{3m}{8}\sin 2(\theta' - \nu)$,

$$\iota = \text{const.} + \frac{3m^2}{8}\cos 2(\theta' - \nu);$$

* In Arts. (12), (8) and (9), in the expressions for $\frac{d\nu}{dt}$, $\sin\iota$ is put by mistake for $\tan\iota$.

and these values, substituted in the equation

$$\sin \lambda = \sin \iota \sin (\theta - \nu),$$

give the moon's latitude.

25. We shall conclude by pointing out briefly how the parallactic inequality and the more correct values of the mean motions of the perigee and node are obtained.

In Art. (16) the term of R, involving the third power of z or $\frac{r}{r'}$, is

$$\frac{m'r^3}{r'^4} \cdot \frac{5p^3 - 3p}{2}.$$

In this term neglect the eccentricities altogether, put for $\frac{5p^3 - 3p}{2}$ its value $\frac{5}{8} \cos 3 (\theta - \theta') + \frac{3}{8} \cos (\theta - \theta')$, retain only the part multiplied by $\cos (\theta - \theta')$, which rises in integration, and, so far as this term of R is concerned, we find

$$\frac{dR}{dr} = \frac{3m'r^2}{r'^4} \left\{\frac{3}{8} \cos (\theta - \theta')\right\},$$

$$\frac{dR}{d\theta} = \frac{m'r^3}{r'^4} \left\{-\frac{3}{8} \sin (\theta - \theta')\right\};$$

$$\text{and} \therefore \quad -\frac{r^2}{h^2} \frac{dR}{dr} = -\frac{9}{8} \frac{m^2}{a} \cdot \frac{a}{a'} \cos (\theta - \theta') \ldots\ldots (A),$$

$$2r^2 \frac{dR}{d\theta} = -\frac{3}{4} a\mu m^2 \cdot \frac{a}{a'} \sin (\theta - \theta');$$

hence in $\frac{\mu}{h^2}$ there arises the term

$$-\frac{3}{4} \frac{m^2}{a} \cdot \frac{a}{a'} \cos (\theta - \theta') \ldots\ldots\ldots\ldots (B);$$

and, observing that e is zero and therefore $\frac{dr}{d\theta} = 0$, in the equation (3) there arises the term

$$(A) + (B) \quad \text{or} \quad -\frac{15}{8} \frac{m^2}{a} \cdot \frac{a}{a'} \cos (\theta - \theta').$$

If, therefore, $F \cos (\theta - \theta')$ be the corresponding term in u, we have

$$F\{c^2 - (1 - m)^2\} = -\frac{15}{8} \frac{m^2}{a} \frac{a}{a'},$$

and therefore

$$F = -\frac{15}{16} \frac{m}{a} \cdot \frac{a}{a'}.$$

Hence we have

$$u = u_1 \left\{ \ldots\ldots - \frac{15}{16} m \frac{a}{a'} \cos(\theta - \theta') \right\}.$$

The term here obtained is the parallactic inequality. (See Airy's *Tracts.*)

26. To obtain the more correct value of the mean motion of the perigee, we must look out in the second member of equation (3) for terms of the form $Ae \cos(\theta - \omega)$. Now it is easy to see that, when we substitute for r, r' and θ' their values to the second order (see Art. 21), in order to expand the second member of (3) to the fourth order, terms of the form $Ae \cos(\theta - \omega)$ will arise only from the combination of the quantities me cos (or sin) $(\theta - 2\theta' + \omega)$ and cos (or sin) $2(\theta - \theta')$. Hence, confining our attention only to the term $me \cos(\theta - 2\theta' + \omega)$, *i.e.* putting for r

$$a \left\{ 1 - \frac{15me}{8} \cos(\theta - 2\theta' + \omega) \right\},$$

and for r' and θ', a and $m\theta + \beta$ simply, we find (see Art. 17)

$$\frac{d(h^2)}{d\theta} = -3a\mu m^2 \left\{ 1 - 4 . \frac{15me}{8} \cos(\theta - 2\theta' + \omega) \right\} \sin 2(\theta - \theta')$$

$$= -3a\mu m^2 \left\{ -\frac{15me}{4} \sin(\theta - \omega) \ldots\ldots \right\},$$

$$h^2 = a\mu \left\{ \ldots\ldots - \frac{3 . 15}{4} m^3 e \cos(\theta - \omega) \ldots \right\};$$

hence, and putting $au - 1$ for $e \cos(\theta - \omega)$, we find

$$\frac{\mu}{h^2} = \ldots\ldots + \frac{3 . 15}{4} m^3 u \ldots\ldots (A).$$

Again, we have (see Art. 18)

$$-\frac{r^2}{h^2}\frac{dR}{dr} = -\frac{m^2}{a} \left\{ 1 - 3 . \frac{15me}{8} \cos(\theta - 2\theta' + \omega) \right\} \frac{3 \cos 2(\theta - \theta')}{2}$$

$$= \frac{m^2}{a} \left\{ \ldots\ldots \frac{9 . 15}{32} me \cos(\theta - \omega) \ldots\ldots \right\}$$

$$= \ldots\ldots\ldots \frac{9 . 15}{32} m^3 u \ldots\ldots\ldots (B).$$

Lastly, we have (see Art. 19)

$$\frac{dr}{d\theta} = a \left\{ \frac{15me}{8} \sin(\theta - 2\theta' + \omega) \right\},$$

and $\therefore\ \dfrac{1}{h^2}\dfrac{dr}{d\theta}\dfrac{dR}{d\theta} = -\dfrac{3m^2}{2a}\cdot\dfrac{15me}{8}\sin(\theta - 2\theta' + \omega)\sin 2(\theta - \theta')$

$$= \ldots\ldots - \frac{3.15}{32}\cdot\frac{m^3 e}{a}\cos(\theta - \omega)\ldots\ldots$$

$$= \ldots\ldots - \frac{3.15}{32}\, m^3 u \ldots\ldots\ldots\ldots\ldots\ldots(C).$$

Hence, adding (A), (B) and (C), we have a term

$$\frac{3.15}{4}\left\{1 + \frac{3}{8} - \frac{1}{8}\right\} m^3 u = \frac{225}{16}\, m^3 u.$$

Hence the differential equation for u (see Art. 20) becomes

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{3m^2}{2}\right) u = \ldots\ldots\ \frac{225}{16}\, m^3 u \ldots\ldots$$

$$\text{or}\quad \frac{d^2u}{d\theta^2} + \left(1 - \frac{3m^2}{2} - \frac{225}{16}\, m^3\right) u = \ldots\ldots\ldots$$

Hence we have $\quad c^2 = 1 - \dfrac{3m^2}{2} - \dfrac{225}{16}\, m^3,$

$$c = 1 - \frac{3m^2}{4} - \frac{225}{32}\, m^3,$$

which determines the more accurate value of the mean motion of the perigee. (See Airy's *Tracts.*)

27. The more accurate value of the mean motion of the node is easily obtained from the equation (see Art. 24)

$$\frac{d\nu}{d\theta} = -\frac{3m^2}{4}\,\{1 - \cos 2(\theta' - \nu)\}$$

by putting for ν, in the second member, its first approximate value

$$\nu = \nu_1 + \frac{3m}{8}\sin 2(\theta' - \nu_1),$$

which, expanding $\cos 2(\theta' - \nu)$, and retaining only the constant part, gives

$$\frac{d\nu}{d\theta} = -\frac{3m^2}{4}\left\{1 - \cos 2(\theta' - \nu_1) - \frac{3m}{4}\sin^2 2(\theta' - \nu_1)\right\}$$

$$= -\frac{3m^2}{4} + \frac{9m^3}{32},$$

which determines the more accurate value of the mean motion of the node. (See Airy's *Tracts.*) M. O. B.

II.—ON THE EQUATION $(D+a)^n\ y = X$.

This short paper is intended to illustrate the formation of all the differential equations connected with an equation of the n^{th} order, namely, its successive primitives of the $(n-1)^{\text{th}}$, $(n-2)^{\text{th}}$, &c. orders. One of the greatest encouragements in the use of the calculus of operations is the perfect view which it gives of all the primitives of a linear equation of the form $(D-a)(D-b)\ldots\ldots y = X$, allowing any one of them to be singled out and formed independently of the rest. This is not so easily done when two or more of the set a, b, &c. are equal, the reason being that some of the primitives in such a case are not linear *with constant coefficients.* I do not intend to consider all the combinations of primitives which may occur in an equation of the general form $(D+a)^n (D+b)^p\ldots\ldots y = X$, but only the particular form $(D+a)^n\ y = X$.

To take first the simplest form, let $a = 0$, or $y^{(n)} = X$. The discussion of this easy case at length will allow of brevity in treating the more general case. First, it is desirable to express $\int(dx)^n\phi x$ in terms of $\int x^n\phi x dx$: let this last be P_n, we have then the following theorems:

$$\Gamma(n+1)\int(dx)^{n+1}\phi x = x^n P_0 - nx^{n-1}P_1 + \&\text{c.};$$

$$\int x^m P_n dx = \frac{x^{m+1}P_n - P_{m+n+1}}{m+1},$$

$$\int ABdx = A\int(dx)\,B - A'\int(dx)^2\,B + A''\int(dx)^3\,B - \&\text{c.}$$

From the last theorem (which is an obvious extension of John Bernoulli's) it follows that if A be a rational and integral function of the n^{th} degree, $\int ABdx$ is always integrable when B can be integrated $n+1$ times. Hence $\rho x\,.\,y^{(n)}$ is integrable *per se*, if $\rho x = a_0 + a_1 x + \ldots\ldots + a_{n-1}x^{n-1}$: its integral being $\rho x\,.\,y^{(n-1)} - \rho' x\,.\,y^{(n-2)} + \&\text{c.}$ This must be called a known theorem;* but it certainly is not well known.

The mode of proceeding with $y^{(n)} = X$ is now obvious enough: let our instance be $y^{iv} = X$, and let $\int Xx^n dx = P_n$. Multiply successively by x^0, x^1, x^2, x^3, and integrate; we thus get the four primitives of the third degree, which are (the constants being included in the symbols P_0 &c.)

$$(1)\quad y''' = P_0,$$
$$(2)\quad xy''' - y'' = P_1,$$
$$(3)\quad x^2y''' - 2xy'' + 2y' = P_2,$$
$$(4)\quad x^3y''' - 3x^2y'' + 6xy' - 6y = P_3.$$

* It is a particular case, deducible from the general method of finding integrating factors for linear expressions.

To find the six primitives of the second order, observe that the first side of (1) is integrable, and remains so when multiplied by x or x^2; also that the first side of (2), or the same multiplied by x, is integrable; also that the first side of (3) is integrable. Here are then the processes for finding the six equations required, which are

$$(5)\ \ y'' = xP_0 - P_1,$$
$$(6)\ \ xy'' - y' = \tfrac{1}{2}(x^2P_0 - P_2),$$
$$(7)\ \ x^2y'' - 2xy' + 2y = \tfrac{1}{3}(x^3P_0 - P_3),$$
$$(8)\ \ xy'' - 2y' = xP_1 - P_2,$$
$$(9)\ \ x^2y'' - 3xy' + 3y = \tfrac{1}{2}(x^2P_1 - P_3),$$
$$(10)\ \ x^2y'' - 4xy' + 6y = xP_2 - P_3.$$

To find the four primitives of the first order, observe that (5) is integrable, and remains so when multiplied by x, and that (6) and (8) are integrable. The four equations required are

$$(11)\ \ y' = \tfrac{1}{2}(x^2P_0 - 2xP_1 + P_2),$$
$$(12)\ \ xy' - y = \tfrac{1}{3}x^3P_0 - \tfrac{1}{2}x^2P_1 + \tfrac{1}{6}P_3,$$
$$(13)\ \ xy' - 2y = \tfrac{1}{6}x^3P_0 - \tfrac{1}{2}xP_2 + \tfrac{1}{3}P_3,$$
$$(14)\ \ xy' - 3y = \tfrac{1}{2}x^2P_1 - xP_2 + \tfrac{1}{2}P_3.$$

Finally, integration of (11) gives the primitive of *the order zero,*

$$(15)\ \ y = \frac{1}{2\,.\,3}(x^3P_0 - 3x^2P_1 + 3xP_2 - P_3).$$

To extend this to the case of $(D+a)^ny = X$. Search in the common way for the factor which renders $(D+a)^ny$ integrable *per se;* it will be found to be $\varepsilon^{ax}(c_0 + c_1x + \dots + c_{n-1}x^{n-1})$, say $\varepsilon^{ax}.\rho x$. To find $D^{-1}.\,\varepsilon^{ax}\rho x\ (D+a)^n\ y$, transform it into $D^{-1}\varepsilon^{ax}\rho x\,\varepsilon^{-ax}D^n\varepsilon^{ax}y$ or $D^{-1}(\rho xD^n\varepsilon^{ax}y)$, which, on account of the degree of ρx, appears in finite form thus,

$$(\rho xD^{n-1} - \rho'xD^{n-2} + \dots \pm \rho^{(n-1)}xD^0)\,\varepsilon^{ax}y$$
$$= \varepsilon^{ax}\{\rho x(D+a)^{n-1} - \rho'x(D+a)^{n-2} + \dots \pm \rho^{(n-1)}x(D+a)^0\}\,y.$$

Proceed thus with $(D+a)^n\,y = X$. Let $\int\varepsilon^{ax}x^n\phi x dx$ be signified by P_n. We have then, as before,

$$\Gamma(n+1)(D+a)^{-(n+1)}\phi x = \varepsilon^{-ax}\left(x^nP_0 - nx^{n-1}P_1 + n\frac{n-1}{2}x^{n-2}P_2 - \dots\right)$$

$$\int x^m P_n dx = \frac{x^{m+1}P_n - P_{m+n+1}}{m+1}$$

$$(D+a)^{-1}.AB = A(D+a)^{-1}B - A'(D+a)^{-2}B + A''(D+a)^{-3}B - \dots$$

Though it may be worth while to exhibit these theorems in terms of the operation $D+a$, yet it is not necessary to pro-

ceed further; for $(D+a)^n\, y = X$ is $D^n(\varepsilon^{ax}y) = \varepsilon^{ax}X$, and the rule of proceeding is to solve $y^{(n)} = \varepsilon^{ax}X$, as in the manner above given, and then to write in the several equations $\varepsilon^{ax}y$, $\varepsilon^{ax}(D+a)\,y$, $\varepsilon^{ax}(D+a)^2y$, instead of y, y', y'', &c. Thus, $xy'' - 2y' = x\int Xxdx - \int Xx^2dx$ being one of the primitives of $y^{iv} = X$, it follows that

$$x\varepsilon^{ax}(D+a)^2\,y - 2\varepsilon^{ax}(D+a)\,y = x\int X\varepsilon^{ax}xdx - \int X\varepsilon^{ax}x^2dx,$$

or $xy''+(2ax-2)\,y'+(a^2x-2a)\,y = \varepsilon^{-ax}x\int X\varepsilon^{ax}xdx - \varepsilon^{-ax}\int X\varepsilon^{ax}x^2dx$, is one of the primitives of

$$y^{iv} + 4ay''' + 6a^2y'' + 4a^3y' + a^4y = X.$$

The following theorem corresponds to that of integration by parts $(D+a)^{-1}\{A\,(D+a)\,B\} = AB - (D+a)^{-1}(BDA)$, which is an immediate consequence of the simple symbolical relation

$$\frac{D}{D+a} = 1 - \frac{a}{D+a}.$$

A. D. M.

III.—ELEMENTARY DEMONSTRATION OF DUPIN'S THEOREM.

If there be three series of surfaces, such that all the surfaces of each series cut the surfaces of the other two series at right angles, the theorem to be proved is that the lines of intersection of any one of the surfaces of the three series, with the surfaces of the two conjugate series, are its lines of curvature.

Let O be any point in which three conjugate surfaces intersect, and let the rectangular axes OX, OY, OZ be perpendicular to the tangent planes of the three surfaces at O. Let

$$f(x, y, z) = \lambda \;\ldots\ldots\ldots\ldots\; (a)$$
$$f_1(x, y, z) = \lambda_1 \;\ldots\ldots\ldots\ldots\; (a_1)$$
$$f_2(x, y, z) = \lambda_2 \;\ldots\ldots\ldots\ldots\; (a_2)$$

be the equations of the three series; and, when proper values are attached to $\lambda, \lambda_1, \lambda_2$, let (a) be the surface touched by YOZ, (a_1) by ZOX, and (a_2) by XOY.

Hence, when $x = 0$, $y = 0$, $z = 0$, we have

$$\left.\begin{aligned} \frac{df}{dy} &= 0, & \frac{df}{dz} &= 0 \\ \frac{df_1}{dz} &= 0, & \frac{df_1}{dx} &= 0 \\ \frac{df_2}{dx} &= 0, & \frac{df_2}{dy} &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (b).$$

Now, since the system is orthogonal,

$$\left.\begin{aligned}\frac{df_1}{dx}\frac{df_2}{dx}+\frac{df_1}{dy}\frac{df_2}{dy}+\frac{df_1}{dz}\frac{df_2}{dz}=0\\ \frac{df_2}{dx}\frac{df}{dx}+\frac{df_2}{dy}\frac{df}{dy}+\frac{df_2}{dz}\frac{df}{dz}=0\\ \frac{df}{dx}\frac{df_1}{dx}+\frac{df}{dy}\frac{df_1}{dy}+\frac{df}{dz}\frac{df_1}{dz}=0\end{aligned}\right\}\ldots\ldots(c).$$

Differentiating the first of these equations with respect to x, the second with respect to y, and the third with respect to z; putting x, y, and z each $= 0$, in the result, and making use of equations (b), we have

$$\left(\frac{df_1}{dy}\right)\left(\frac{d^2f_2}{dxdy}\right)+\left(\frac{df_2}{dz}\right)\left(\frac{d^2f_1}{dzdx}\right)=0,$$

$$\left(\frac{df}{dx}\right)\left(\frac{d^2f_2}{dxdy}\right)+\left(\frac{df_2}{dz}\right)\left(\frac{d^2f}{dydz}\right)=0,$$

$$\left(\frac{df}{dx}\right)\left(\frac{d^2f_1}{dxdz}\right)+\left(\frac{df_1}{dy}\right)\left(\frac{d^2f}{dydz}\right)=0,$$

the brackets denoting that in the quantities enclosed, x, y, z are put $= 0$. From these equations we conclude that

$$\left(\frac{d^2f}{dydz}\right),\quad \left(\frac{d^2f_1}{dzdx}\right),\quad \left(\frac{d^2f_2}{dxdy}\right) \text{ are each } = 0.$$

Now, since YOZ is the tangent plane to the surface (a) at O, we have

$$x=\frac{1}{2}\left\{\left(\frac{d^2x}{dy^2}\right)y^2+2\left(\frac{d^2x}{dydz}\right)yz+\left(\frac{d^2x}{dz^2}\right)z^2\right\}+\&c.\ldots.(d)$$

for points in the surface adjacent to O. To determine $\left(\frac{d^2x}{dydz}\right)$, we have, from ($a$),

$$\frac{dx}{dy}=-\frac{\dfrac{df}{dy}}{\dfrac{df}{dx}}.$$

Hence, differentiating with regard to z, and putting x, y, and z each $= 0$, in the result,

$$\left(\frac{d^2x}{dydz}\right)=-\frac{\left(\dfrac{d^2f}{dydz}\right)}{\left(\dfrac{df}{dx}\right)}=0.$$

Hence (d) becomes

$$x = \frac{1}{2}\left\{\left(\frac{d^2x}{dy^2}\right)y^2 + \left(\frac{d^2x}{dz^2}\right)z^2\right\} + \&c.$$

Hence the planes of xy and xz contain the principal sections of (a) through O, and therefore the lines of intersection of (a_1) and (a_2) with (a), touch the principal sections of (a) at O. Now O may be any point in one of the surfaces (a), and therefore each of these surfaces has its lines of curvature traced upon it by the surfaces of the series (a_1), (a_2). Similarly it may be shewn that each surface, (a_1) has its lines of curvature traced by (a_2) and (a), and each surface (a_2) by (a) and (a_1), which is the theorem to be proved.

By differentiating the first of equations (c) with regard to y and z respectively, and putting x, y, and z each $= 0$ in the results, and performing corresponding operations on the second and third, we obtain expressions for

$$\left(\frac{d^2f_1}{dx^2}\right), \left(\frac{d^2f_2}{dx^2}\right), \left(\frac{d^2f}{dy^2}\right), \left(\frac{d^2f_2}{dy^2}\right), \left(\frac{d^2f}{dz^2}\right), \left(\frac{d^2f_1}{dz^2}\right),$$

in terms of $$\left(\frac{d^2f}{dx^2}\right), \left(\frac{d^2f_1}{dy^2}\right), \left(\frac{d^2f_2}{dz^2}\right),$$

which lead directly to the expression for the curvatures of the principal sections of the three surfaces at O, given by Lamé, in his *Memoir on Curvilinear Co-ordinates.*

P. Q. R.

IV.—NOTE ON A DEFINITE MULTIPLE INTEGRAL.

In the XVIII[th] number of this *Journal* Mr. Boole pointed out the incorrectness of a theorem given by M. Catalan. The following pages contain a brief demonstration of the result to which he was led. Both he and M. Catalan made use of what is generally known as Liouville's theorem, and thus perhaps rendered their analysis less simple than it would otherwise have been.

Let us transform the integral

$$\int dx_1 \ldots \int dx_n f(a_1x_1 + \ldots + a_nx_n)$$

by the assumption

$$a_1x_1 + \ldots a_nx_n = \alpha u_1 \ldots (\alpha^2 = \Sigma a^2),$$

and by $(n-1)$ other linear relations connecting $x_1 \ldots x_n$ and $u_1 \ldots u_n$, and such that

$$\Sigma x^2 = \Sigma u^2.$$

Then, as is well known, $dx_1 \ldots dx_n$ is to be replaced by $du_1 \ldots du_n$, and thus

$$\int dx_1 \cdots \int dx_n f(a_1x_1 + \cdots + a_nx_n) = \int du_1 f \alpha u_1 \int du_2 \cdots \int du_n \cdots (1).$$

Let the integrations on the first side of this equation include all values of the variables which do not transgress the limits

$$\Sigma x^2 = A^2, \quad \Sigma x^2 = B^2,$$

where B is supposed to be greater than A. Then, as $\Sigma x^2 = \Sigma u^2$, the corresponding limits on the second side of the equation are

$$\Sigma u^2 = A^2, \quad \Sigma u^2 = B^2.$$

Transform the integral in x by assuming

$$x_1 = r\cos\theta_1, \quad x_2 = r\sin\theta_1\cos\theta_2 \ldots \quad x_n = r\sin\theta_1\ldots\sin\theta_{n-1};$$

and that in u, by similar assumptions,

$$u_1 = r\cos\theta'_1 \quad u_2 = r\sin\theta'_1\cos\theta'_2 \ldots u_n = r\sin\theta'_1\ldots\sin\theta'_{n-1}.$$

I may be allowed to mention that this transformation, which appears to have been given for the first time by Mr. Boole, in the last number of the *Journal*, had occurred to me before I had seen his paper. His analysis leads at once to the conclusion that $dx_1\ldots dx_n$ is to be replaced by

$$r^{n-1}\sin^{n-2}\theta_1\ldots\sin\theta_{n-2} \quad dr d\theta_1\ldots d\theta_{n-1}.$$

This may be also proved by successive substitutions in the manner pointed out in the case of three variables by Mr. A. Smith, in the first volume of the *Journal*.

Thus (1) becomes, since $\Sigma x^2 = \Sigma u^2 = r^2$,

$$\int_A^B r^{n-1}dr \int \sin^{n-2}\theta_1 d\theta_1 \ldots \int d\theta_{n-1} f\{r(a_1\cos\theta_1 + \ldots a_n\sin\theta_1\ldots\sin\theta_{n-1})\}$$
$$= \int_A^B r^{n-1}dr \int \sin^{n-2}\theta'_1 f(ar\cos\theta'_1)\, d\theta'_1 \int \sin^{n-3}\theta'_2 d\theta'_2 \ldots$$
$$\int \sin\theta'_{n-2} d\theta'_{n-2} \int d'\theta_{n-1}.$$

The limits for θ and θ' are the same.

That this equation may subsist for all values of A and B, it is necessary and sufficient that

$$\int \sin^{n-2}\theta_1 d\theta_1 \ldots \int d\theta_{n-1} f\{r(a_1\cos\theta_1 + \ldots a_n\sin\theta_1 \ldots \sin\theta_{n-1})\}$$
$$= \int \sin^{n-2}\theta'_1 f(ar\cos\theta'_1)\, d\theta'_1 \int \sin^{n-3}\theta'_2 d\theta'_2 \ldots \int d\theta'_{n-1} \ldots\ldots (2).$$

With respect to the limits of θ and θ', it is not difficult to perceive that if $\theta_1\ldots\theta_{n-1}$ are taken between the limits 0 and π, $x_1\ldots x_{n-1}$ will receive all the values of which they are capable, namely, all that included between $-r$ and $+r$; and that the same set of values cannot occur more than once. But in order that x_n may vary from 0 to $-r$, it is necessary to extend the superior limit of θ_{n-1} from π to 2π. Thus the limits of θ_{n-1} are 0 and 2π, while those of the other variables $\theta_1\ldots\theta_{n-2}$ are 0 and π. And similarly for θ'.

On the second side of (2) we have the factor

$$\int_0^\pi \sin^{n-3}\theta'_2 d\theta'_2 \int_0^\pi \sin^{n-4}\theta'_3 d\theta'_3 \ldots \int_0^{2\pi} d\theta'_{n-1} \ldots (A).$$

Let n be odd, then we shall have

$$\int_0^\pi \sin^{n-3}\theta'_2 d\theta'_2 \int_0^\pi \sin^{n-4}\theta'_3 d\theta'_3$$

$$= \frac{(n-4)\ldots 3 \,.\, 1}{(n-3)\ldots 4 \,.\, 2} \cdot \frac{(n-3)\ldots 4 \,.\, 2}{(n-4)\ldots 3 \,.\, 1} 2\pi = \frac{2\pi}{n-3}.$$

Similarly

$$\int_0^\pi \sin^{n-5}\theta'_4 d\theta'_4 \int_0^\pi \sin^{n-6}\theta'_5 d\theta'_5 = \frac{2\pi}{n-5}, \quad \&c. = \&c.$$

Lastly $$\int_0^{2\pi} d\theta'_{n-1} = 2\pi.$$

Thus $$(A) = 2\frac{\pi^{\frac{n-1}{2}}}{\left(\frac{n-1}{2} - 1\right)\left(\frac{n-1}{2} - 2\right)\ldots 2 \,.\, 1} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}.$$

Again, if n be even, we get

$$(A) = 2\frac{\pi^{\frac{n-2}{2}}}{\left(\frac{n-1}{2} - 1\right)\left(\frac{n-1}{2} - 2\right)\ldots \frac{3}{2}\frac{1}{2}}.$$

And this, multiplied by $\frac{\sqrt{(\pi)}}{\Gamma(\frac{1}{2})}$, or unity, gives, as before,

$$(A) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)};$$

and thus (2) becomes, making $r = 1$,

$$\int_0^\pi \sin^{n-2}\theta_1 d\theta_1 \int_0^\pi \sin^{n-3}\theta_2 d\theta_2 \ldots \int_0^{2\pi} d\theta_{n-1} f(a_1 \cos\theta_1 + \ldots \&c.)$$

$$= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi \sin^{n-2}\theta'_1 d\theta'_1 f(a\cos\theta'_1) \ldots (3).$$

This is the general result given at the end of Mr. Boole's paper, and which includes the others. R. L. E.

IV.—NOTE ON SOME POINTS IN THE THEORY OF HEAT.

In problems relative to the motion of heat in solid bodies, the initial distribution, which is entirely arbitrary, is usually one of the data. When this is the case, and the circumstances in which the body is placed are known, the distribution at any subsequent period is fully determined, and if our analysis had sufficient power, would become known in every case. The solution of the problem would be an expression for the temperature of any point in the body in terms of the co-ordinates of the point, and the times measured from the instant at which the distribution is given.

It is in many cases an interesting investigation to examine what this expression becomes when negative values are assigned to the time. If, for a particular negative value $-\tau$, we find that the expression gives an actual arithmetical value for the temperature of every point in the body, the distribution represented will be such that, if it had existed at a time τ before the initial instant, the given initial distribution would have been produced by the spontaneous motion of the heat.

It is clear, however, that the arbitrary initial distribution may be of such a nature that it cannot be the natural result of any previous possible distribution, or that it cannot be any stage except the first in a system of varying temperatures. This, for instance, will be the case if there be any abrupt transitions in the initial temperatures of adjacent points, or if the curves representing the initial temperatures of points situated along any straight line through the body, have cusps or angular points; for, though we may suppose such a distribution to be arbitrarily made, it could obviously not be produced by the spontaneous motion of heat from any other preceding distribution, and all such abrupt transitions or angles which may exist in an initial distribution, will disappear instantaneously after the motion has commenced.

If, however, in any case, a complete solution of the problem has been obtained, that is, if an expression for the temperature of any point, in terms of its co-ordinates, and the time has been found, this expression must assume some *form*, when negative values are given to the time; and it is my object here to examine the nature of this form, when the initial state is such as not to be deducible from a previous distribution.

The simplest case is that of the linear motion of heat, and as from it we are enabled to understand the nature of the

problem in its most general form, I shall, for the present, confine myself to this case. The case of a thin rod, protected from any lateral radiation, or of an infinite solid, in which the temperature is distributed in parallel isothermal planes, is what is usually contemplated in speaking of the linear motion of heat; but the case of a thin rod losing heat by radiation from its sides may also be readily reduced to the same as the two former. For simplicity, however, we may suppose that there is no lateral radiation.

Let a fixed point O in the rod be considered as origin, let the distance to any point P in the rod be x, and let the temperature of P, at the time t from the initial instant, be v. The equation of the motion of the heat will be

$$\frac{dv}{dt}=\frac{d^2v}{dx^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

if, for brevity, we choose the unit of heat such that the conducting power of the body, referred to a unit of its volume, may be unity.

The complete integral of this equation is

$$v = \Sigma A_i \varepsilon^{-m_i^2 t} \cos(m_i x + n_i) \ldots\ldots\ldots\ldots (2).$$

Let ${}_0v$ be the initial temperature of P. Then, putting $t=0$, in (2), we have

$${}_0v = \Sigma A_i \cos(m_i x + n_i) \ldots\ldots\ldots\ldots (3).$$

Now, whatever be the arbitrary nature of the function ${}_0v$, whether continuous or discontinuous, it has been shewn by Fourier that it may be represented by a series such as the second member of (3), and he has shewn how, when the value of ${}_0v$, corresponding to any value of x, is given, the constants A_i, m_i, n_i may be determined. The series thus found is an actual quantitative representation of ${}_0v$, and is necessarily convergent. In fact, the proof of its convergence in every case must be included in the demonstration of the possibility of representing any function ${}_0v$ by the series (3).

Now the expression for v at time t, is found by multiplying the respective terms of the second members of (3), by the quantities

$$\varepsilon^{-m_1^2 t},\quad \varepsilon^{-m_2^2 t},\ \&\text{c.} \ldots\ldots\ldots\ldots\ldots (a);$$

a series which converges when t is positive, (m_1, m_2, &c. being arranged in ascending order of magnitude). Hence, when t is positive, the series for v is still more convergent than the series for ${}_0v$, and therefore (2) gives a convergent series for v for every positive value of t. If, however, the time considered be τ *before* the initial instant, the series (a) will become

$$\varepsilon^{m_1^2 \tau},\quad \varepsilon^{m_2^2 \tau},\ \&\text{c.} \ldots\ldots\ldots\ldots\ldots (b),$$

which is divergent. Hence it will depend on the degree of the convergence of the series for ${}_0v$, whether the expression of v in this case be convergent or divergent. In the latter case the distribution represented will be impossible, and therefore there will be no distribution from which the initial distribution can be derived, by spontaneous motion, in the time τ. There are three cases of the initial distribution, which we must specially consider.

1. If the convergence of the expression for ${}_0v$, or of the series of coefficients

$$A_1, \ A_2, \ \&c.$$

be of a *lower order* than that of the series (a); that is, if the ultimate convergence of this series be less than that of (a), for every positive value of t, then for any finite value, however small, of τ, the series

$$A_1\varepsilon^{m_1^2\tau}, \quad A_2\varepsilon^{m_2^2\tau}, \quad \&c.$$

will be ultimately divergent. Hence, in this case, for any time, however small, before the initial instant, the distribution will be impossible, and therefore the given distribution cannot be any stage but the first in a system of varying temperatures.

2. If the convergence of the series for ${}_0v$ be ultimately of the *same order* as that of the series (a); that is, if the coefficients A_1, A_2, &c. can be put under the forms

$$a_1\varepsilon^{-m_1^2\tau'}, \quad a_2\varepsilon^{-m_2^2\tau'}, \quad \&c.,$$

where a_1, a_2, &c. form a series which ultimately has a convergence of a *lower order* than that of the series (a), according to the definition previously given; then, at a time τ' before the initial instant, the distribution is represented by

$$v' = \Sigma a_i \cos (m_i x + n_i),$$

which belongs to the first class considered, and is therefore an essentially primitive distribution. Hence, in this case, a finite *age*, τ' may be assigned to the given initial distribution.

3. If the convergence of the series for ${}_0v$ be of a *higher order* than that of the series (a); or if, for any finite value of τ, however great, the series

$$A_1\varepsilon^{m_1^2\tau}, \quad A_2\varepsilon^{m_2^2\tau}, \quad \&c.$$

is ultimately convergent; then, for every finite negative value of t, the series for v will converge, and therefore v will have a possible expression. Hence, in this case, no limit can be assigned to the *age* of the initial distribution.

The simplest case of distributions which belong to the third class, is that in which the expression for ${}_0v$ is composed of a finite number of terms of the form

$$A \cos (mx + n);$$

but we may also readily form series of an infinite number of terms, which will be sufficiently convergent to satisfy the condition stated for the third class, and therefore any initial distribution represented by such a series may be deduced from distributions existing previously for any length of time, however great.

These remarks are sufficient to indicate the *nature* of the question proposed. The details of the convergence or divergence of the series employed will depend on the values which must be assigned to m_1, m_2, &c. for satisfying the conditions of any particular problem.

If, instead of the initial distribution of heat in the rod, from $-\infty$ to $+\infty$ being given, we have only a part, that for instance on the positive portion of the rod, we must, to make the problem determinate, have some other condition given. In such cases, the part of the rod, over which the initial distribution is not given may be conceived to be removed, and the second condition will then generally relate to the extremity of the part of the rod considered. In a previous paper (*On the Linear Motion of Heat*, vol. III. p. 170), I have shewn how a solution of the problem may be obtained by determining the initial distribution which must be made on the negative part of the rod, in order that the condition relative to the zero point may be fulfilled. Thus, if the temperature at the zero point, or the extremity of the part of the rod originally given, be constrained to be a given arbitrary function of the time, this constraint may be effected by producing the rod indefinitely in the negative direction, and impressing on the part produced a certain initial distribution, determined by equations (6) and (10) of the Article already referred to, in terms of the arbitrary function of the time, and the given initial distribution on the positive side. In many cases, however, the distribution so determined will be impossible, either over the whole extent of the negative part of the rod, or from some point at a finite distance on the negative side, to an infinite distance. Notwithstanding this impossibility, the solution of the problem obtained by the method described above will still give a finite possible expression for the temperature of any point on the positive side, at any time subsequent to the initial instant, which will be the tempera-

ture that would be actually assumed by the point, if the temperature of the zero point had, by some external application, been constrained to satisfy the given condition; and the only thing indicated by the impossibility of the distribution on the negative side will be that this constraint cannot be effected actually, by adding the negative part of the rod, with a certain initial distribution of heat, to the given positive part. It is unnecessary to enter into this question separately here, as the details are very similar to those given above. It should be remarked however that the possibility or impossibility of the distribution on the negative part will depend entirely on the function of the time, which expresses the variable temperature of the zero point, and not at all on the given initial distribution on the positive part; and on this account, if the initial distribution on the negative part be impossible, all its subsequent forms will generally be impossible also.

Before leaving this subject, it may be well to notice a point relative to the theory of isothermal surfaces, or surfaces of equilibrium, which involves considerations analogous to those with which we have been occupied above.

It is a known theorem that an attraction every where perpendicular to any given closed surface S, may be produced by the distribution of a given quantity of matter m over the surface, according to a law which is in every instance determinate; but in general it will be impossible to produce the same effect by the distribution of matter over any surface in the interior of S, not coinciding with it. If, for instance, S were the surface of a cube, or any surface containing points or edges, this would obviously be impossible, and it would probably also be impossible in the case considered by Poisson, in which S is composed of two spherical surfaces. In every case there will be an infinite series of surfaces of equilibrium without S, becoming ultimately a series of spheres having the centre of gravity of m for their common centre, each of which is such that m exerts an attraction on any point in it in the direction of the normal. Hence, if one of these surfaces, s, be given, we may not only produce an attraction every where perpendicular to it, by matter distributed over itself, but by matter distributed over S, or over any surface between S and s, enclosing the former. Hence s is analogous to a distribution of heat which may be produced by a previously existing distribution. But, unless S itself have the particular property we are considering in s, we cannot produce an attraction perpendicular to s by any distribution on

a surface within S. Thus s is analogous to a distribution of heat which cannot be produced by any previous distribution existing at a time before it greater than a certain limit τ.

Again, in some cases the series of surfaces of equilibrium, of which s is one, may be continued indefinitely inwards, till we arrive at a surface enclosing no space, as for instance when s is an ellipsoid, in which case the surfaces of equilibrium are confocal ellipsoids, and the series within s may be continued till we arrive at an elliptical disc. Such a case is analogous to that of a distribution of heat, which may be produced from a distribution existing an unlimited time before.

P. Q. R.

V.—ON THE INTENSITY OF LIGHT IN THE SHADOW OF A VERY SMALL CIRCULAR DISK.

It has been proved experimentally by M. Arago, that the central point of the shadow of a very small opaque circular disk is sensibly as bright as if the light were not intercepted. Fresnel has given a general explanation of this curious result in the Appendix to his *Memoir on the Undulatory Theory of Light*, (*Mem. de l'Institut.*, tom. v. p. 460. *Ann.* 1821–22), and states that Poisson has proved the same thing by direct integration; but he gives no hint as to where this proof can be found.

The usual method of finding the illumination of a point appears in this instance to present some difficulty. For if (fig. 1) A be the centre of the disk, Q the point in the shadow whose illumination is to be found, and we suppose the waves of light plane and parallel to the plane of the disk, also if $PQ = u$, it is easily seen that the whole vibration at Q will be expressed by

$$2\pi C \int \sin \frac{2\pi}{\lambda} (vt - u) . du.$$

The superior limit being ∞, and the inferior Qa, a being a point on the rim of the disk.

The integration introduces at the superior limit the rather doubtful expression $\cos \infty$: but, supposing this to vanish, as the nature of the case shews that it ought, for the vibration at Q cannot be affected by the small waves originating at very distant points of the general incident plane wave, still there is another difficulty; for the integration above indicated, if performed, gives

$$C\lambda \cos \frac{2\pi}{\lambda} (vt - Qa)$$

for the total vibration at Q, and therefore the illumination

will be expressed by $C^2\lambda^2$, which expression being independent of the radius of disk, is considered to prove the proposition. But the difficulty is, that this expression is true however large the disk may be, as well as when it is very small, which is manifestly absurd. It is true that the same reasons which led us to neglect the quantity cos ∞ would teach us to neglect the expression $\cos \frac{2\pi}{\lambda}(vt - Qa)$, when the radius of the disk is very large, and so the illumination would become zero, as it ought: but this does not remove the difficulty from the analysis; for, according to the principle usually adopted, the illumination ought *not* to vanish, when the radius of the disk is considerable; because, since, when a general wave is broken up, the effect due to the small elementary wave is supposed to depend on the distance of the illuminated point from the origin of the small wave, and on that only, it follows that the effect of a small wave spreading from the rim of a disk whose radius is AP (fig. 2) in exciting a vibration at the point Q, will be the same as that of a wave spreading from the rim of a disk whose radius is AP' in exciting a vibration at Q', provided only that $PQ = P'Q'$. In fact the common method does not take into account the effect of *obliquity*, and seems to involve the same error as if, in calculating the illumination of a point by a bright surface according to the principles of common Optics, we were to omit the consideration of the fact that the quantity of light emanating in any direction varies as the sine of the angle of emanation.

This problem has been very elegantly treated in the *Mathematical Journal* (vol. II. p. 141), and the result there obtained is the same as that found above, viz. that the illumination is measured by $C^2\lambda^2$: but the method there given seems to labour under precisely those difficulties which have been just alluded to. The fact seems to be, that the principle of Huyghens, which supposes the illumination of a point on which a luminiferous undulation is incident, to be the same as that which would result from the combination of spherical wavelets diverging with equal intensity in all directions from the elements of the wave-front, cannot be looked upon as a physical principle, but only as an artifice rendered necessary by the state of analysis, and which will not always represent the physical conditions of the problem.

In this paper I have adopted Huyghens' principle so far as this, that I consider the vibration of the ether at a given

point to be the sum of the vibrations which would exist if the wave-front were divided into elements and each of these elements were a centre of disturbance; but I do not suppose the wavelets to diverge spherically, but according to another law which I shall investigate.

With this view I shall first find an integral of the partial differential equation, which seems applicable to the case of light, in the particular case *in which the vibration is symmetrical with respect to a certain axis, and the effect rapidly diminishes as we recede from that axis.*

I shall assume, as the equation applicable to the vibrations of the ether,

$$\frac{d^2\phi}{dt^2} = a^2\left\{\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right\} \dots\dots\dots\dots (1),$$

and let the axis of z be that about which every thing is symmetrical: let $x^2+y^2=\rho$, then ϕ is a function of z, ρ, and t; also

$$\frac{d^2\phi}{dx^2} = 4\frac{d^2\phi}{d\rho^2}x^2 + 2\frac{d\phi}{d\rho},$$

$$\frac{d^2\phi}{dy^2} = 4\frac{d^2\phi}{d\rho^2}y^2 + 2\frac{d\phi}{d\rho};$$

$$\therefore \quad \frac{d^2\phi}{dt^2} = a^2\left\{\frac{d^2\phi}{dz^2} + 4\rho\frac{d^2\phi}{d\rho^2} + 4\frac{d\phi}{d\rho}\right\} \dots\dots (2).$$

Now if ϕ diminishes very rapidly as ρ increases; that is, if we may suppose ρ very small, the term $\rho\dfrac{d^2\phi}{d\rho^2}$ will be negligeable as compared with $\dfrac{d\phi}{d\rho}$; or at all events we may integrate upon that hypothesis, and if it should turn out that this is not the case, the method must be abandoned.

Neglecting, then, the term $\rho\dfrac{d^2\phi}{d\rho^2}$, our equation becomes

$$\frac{d^2\phi}{dt^2} = a^2\left\{\frac{d^2\phi}{dz^2} + 4\frac{d\phi}{d\rho}\right\} \dots\dots\dots\dots (3).$$

Let
$$\phi = u\sin\frac{2\pi}{\lambda}(vt - z),$$

where u involves ρ only: differentiating and substituting, we have

$$-\frac{4\pi^2}{\lambda^2}v^2.u = a^2\left\{-\frac{4\pi^2}{\lambda^2}u + 4\frac{du}{d\rho}\right\},$$

$$\therefore\ u = ae^{-\frac{\pi^2}{\lambda^2}\frac{v^2-a^2}{a^2}\cdot\rho},$$

$$\text{and } \phi = ae^{-\frac{\pi^2}{\lambda^2}\frac{v^2-a^2}{a^2}\rho}\ \sin\frac{2\pi}{\lambda}(vt-z)\ \ldots\ldots(4).$$

Let us now see whether $\rho\dfrac{d^2\phi}{d\rho^2}$ was rightly neglected: $\rho\dfrac{d^2\phi}{d\rho^2}\div\dfrac{d\phi}{d\rho}$ will be small when ρ is small, provided $\dfrac{v^2-a^2}{\lambda^2a^2}$ is not large: and this is a condition which is undoubtedly fulfilled; for, though λ is very small, a is very large indeed, and v^2 also cannot be very different from a^2. I think, therefore, we are justified in considering that the expression (4) very nearly represents the true physical conditions of the problem: in order that it may do so, however, v must be greater than a, and for this I can give no physical reason; I shall, however, assume that it is so, and proceed to apply formula (4) to the question in hand.

If we take an annulus of the plane incident wave, whose radius is $\sqrt{\rho}$, the area of the annulus will be $\pi d\rho$, and if AQ (fig. 1) $= f$, the vibration at Q due to the waves spreading from the different points of this annulus, will be expressed by

$$\pi a\int_{b^2}^{\infty} e^{-\frac{\pi^2}{\lambda^2}\frac{v^2-a^2}{a^2}\rho}\ \sin\frac{2\pi}{\lambda}(vt-f)\,d\rho$$

$$= \pi a\,\frac{\lambda^2}{\pi^2}\cdot\frac{a^2}{v^2-a^2}\cdot e^{-\frac{\pi^2}{\lambda^2}\frac{v^2-a^2}{a^2}b^2}\ \sin\frac{2\pi}{\lambda}(vt-f),$$

where $b =$ the radius of the disk. And therefore the illumination will be expressed by

$$a^2\,\frac{\lambda^4}{\pi^2}\cdot\left(\frac{a^2}{v^2-a^2}\right)^2 e^{-\frac{2\pi^2}{\lambda^2}\frac{v^2-a^2}{a^2}\cdot b^2}.$$

This expression diminishes very rapidly as b increases, but if b be small it approaches very nearly to the value $a^2\,\dfrac{\lambda^4}{\pi^2}\left(\dfrac{a^2}{b^2-a^2}\right)^2$, which is its value if $b = 0$, or if the light was not intercepted. And this seems to agree sufficiently accurately with the experimental result.

I will now apply the same method to express the illumination at any point within the geometrical shadow.

If we take the projection of the given point on the disk as the origin, and call its distance from the centre g, and

measure θ from the line joining this point and the centre, the equation to the circumference of the disk will be

$$r^2 \sin^2 \theta + (r \cos \theta - g)^2 = b^2,$$

or

$$r^2 - 2rg \cos \theta = b^2 - g^2$$

$$r = g \cos \theta + \sqrt{(b^2 - g^2 \sin^2 b)}$$

$$\rho = g^2 \cos 2\theta + b^2 + 2g \cos \theta \sqrt{(b^2 - g^2 \sin^2 \theta)}.$$

If instead of $\frac{\pi^2}{\lambda^2} \frac{v^2 - a^2}{a^2}$ we put n, for shortness, and call the illumination at the point in question I, it will be easily seen that

$$\sqrt{I} = \frac{a}{2} \iint e^{-n\rho} d\theta d\rho,$$

where the limits of ρ are the above value and ∞, and the limits of θ are 0 and π, if we double the integral, since only half the disk is included between 0 and π;

$$\therefore\ \sqrt{I} = \frac{a}{n} \int_0^\pi e^{-n\,[g^2 \cos 2\theta + b^2 + 2g \cos \theta\ d\theta \sqrt{(b^2 - g^2 \sin^2 \theta)}]}$$

$$= \frac{a}{n}\ e^{-nb^2} \int_0^\pi e^{-n\,[g^2 \cos 2\theta + 2g \cos \theta\ d\theta \sqrt{(b^2 - g^2 \sin^2 \theta)}]}.$$

The integration here indicated cannot be performed generally, but there are some particular cases which are worth noticing.

If g be so small that g^2 and higher powers may be neglected, the integral becomes

$$\int_0^\pi e^{-2nbg \cos \theta}\ d\theta = \int_0^\pi (1 - 2nbg \cos \theta)\ d\theta = \pi,$$

or the illumination is the same as at the centre.

If we take in the square of g, the integral becomes

$$\int_0^\pi e^{-ng^2 \cos 2\theta}\ e^{-2nbg \cos \theta}\ d\theta = \int_0^\pi (1 - 2nbg \cos \theta - ng^2 \cos 2\theta + 2n^2b^2g^2 \cos^2 \theta)\ d\theta$$

$$= \pi\ (1 + n^2b^2g^2),$$

$$\therefore\ \ I = \frac{\pi^2 a^2}{n^2} e^{-2nb^2} (1 + 2n^2b^2g^2).$$

Hence, the centre is not so bright as an annulus taken very near to it.

There is one other case which I shall consider, viz. for the geometrical shadow, or when $g = b$; in this case the integral becomes

$$\int_0^\pi e^{-nb^2 (1+2 \cos 2\theta)}\ d\theta = e^{-nb^2} \int_0^\pi e^{-2nb^2 \cos 2\theta}\ d\theta,$$

which is less than π, because e^{-nb^2} is < 1, and

$$\int_0^\pi e^{-2nb^2 \cos 2\theta} \, d\theta = \int_0^\pi (1 - 2nb^2 \cos 2\theta) \, d\theta = \pi, \text{ nearly,}$$

since b is very small. Hence the line of the geometrical shadow will be less bright than the centre. Between these there must be at least one ring of maximum brightness, there may be more.

How far all this agrees with experiment I cannot say, but at least this method of considering the problem appears to embrace the two great facts of the brightness of the centre, and the existence of rings in the shadow.

H. G.

VI.—ON THE MOTION OF THE CENTRE OF GRAVITY OF BROKEN BODIES.

By W. Walton, M.A. Trinity College.

The fundamental principle adopted by Huyghens in the determination of the Centre of Oscillation or Agitation of Complex Pendulums, which forms the subject of the fourth part of the Horologium Oscillatorium, is enunciated in the following words: "Si pendulum e pluribus ponderibus compositum, atque e quiete dimissum, partem quamcunque oscillationis integræ confecerit, atque inde porro intelligantur pondera ejus singula, relicto communi vinculo, celeritates acquisitas sursum convertere, ac quousque possunt ascendere; hoc facto, centrum gravitatis ex omnibus compositæ, ad eandem altitudinem reversum erit, quam ante inceptam oscillationem obtinebat."

From the nature of the argument of this section of the Horologium Oscillatorium, it seems probable, although no where, I believe, expressly stated, and although in opposition to the direct meaning of the words which we have quoted, that Huyghens really supposed each of the disconnected molecules, after arriving at its point of zero velocity, to be there permanently retained; so that the 'centrum gravitatis ex omnibus compositæ' should be considered as permanently assuming its original altitude from the date of the arrival of the molecule of greatest velocity at its maximum altitude. Such a modification of the proposition is at any rate necessary to its truth. The corrected form of the principle which we have mentioned involves indirectly, and in a particular case, the principle of Vis Viva, a generalization which in fact resulted from the principle of Huyghens in the hands of John and Daniel Bernoulli. A demonstration of the principle of Huyghens,

in its unmodified form, has been attempted by Professor Whewell, in a work entitled *The Mechanics of Engineering*, p. 142, by the application of the general principle of Vis ia. The reasoning however of the Professor is not sfactory, in consequence of his having omitted to take account of the idea of time, an essential element of the problem.

The object of this paper is to offer a few remarks on the motion of the centre of gravity of the disconnected molecules, supposing them not to be retained in their respective positions of zero velocity, but to be capable of descending spontaneously from their positions of instantaneous rest. It is not necessary to suppose the body to have been oscillating, before the dissolution of the connections of its molecules, about a fixed axis in the manner of a pendulum, but we may imagine it to have descended from rest in any conceivable way; the remarks we are about to make being generally applicable.

Let m, m', m''.... represent the masses of the molecules, h, h', h''.... their initial distances below the horizontal line through the initial position (G) of their centre of gravity; x, x', x''....their distances below the same line at the moment of the dissolution of their mutual connections; then, if u, u', u''.... denote the velocities of the molecules on the commencement of their independent motions, we have, by the principle of the Conservation of Vis Viva,

$$mu^2 + m'u'^2 + m''u''^2 + \dots = 2g\,\{m(x-h) + m'(x'-h') + m''(x''-h'') + \dots\};$$

but, by the property of the centre of gravity of any number of bodies,

$$mh + m'h' + m''h'' + \dots = 0,$$

and

$$mx + m'x' + m''x'' + \dots = (m + m' + m'' + \dots)\,\bar{x},$$

where $\bar{x}$ denotes the distance of the centre of gravity (H) of the molecules at the moment of their mutual disconnection below the horizontal line through (G); hence

$$mu^2 + m'u'^2 + m''u''^2 + \dots = 2g\,(m + m' + m'' + \dots)\,\bar{x} \dots (1).$$

Again, let v, v', v''.... denote contemporaneous velocities of the molecules in their independent movements, and y, y', y'' ......corresponding contemporaneous altitudes above the horizontal line through (H); also let k, k', k''.... denote the values of y, y', y''.... at the instant of disconnection. We will commence with supposing the molecules to be constrained to move from the moment of detachment along smooth invariable lines, and not to be subject to mutual col-

lision; then, by the theory of the motion of heavy particles, whether the paths of the molecules previous to disconnection be joined continuously with those which they afterwards pursue, or their motions be diverted at the instant of their mutual detachment without loss of velocity into different directions,

$$\left.\begin{array}{l} mv^2 = mu^2 - 2mg\,(y-k), \;\; m'v'^2 = m'u'^2 - 2m'g\,(y'-k'), \\ m''v''^2 = m''u''^2 - 2m''g\,(y''-k''), \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots .(2);$$

we have therefore

$$mv^2 + m'v'^2 + m''v''^2 + \ldots . = mu^2 + m'u'^2 + m''u''^2 + \ldots .$$
$$- 2g\,\{m(y-k) + m'(y'-k') + m''(y''-k'') + \ldots .\};$$

but, by the property of the centre of gravity of bodies,

$$mk + m'k' + m''k'' + \ldots . = 0,$$

and $\quad my + m'y' + m''y'' + \ldots . = (m + m' + m'' + \ldots .)\,\bar{y},$

where $\bar{y}$ denotes the altitude of the centre of gravity of the disconnected molecules at any time above the horizontal line through (H); hence

$$mv^2 + m'v'^2 + m''v''^2 + \ldots .$$
$$= mu^2 + m'u'^2 + m''u''^2 + \ldots - 2g\,(m + m' + m'' + \ldots)\,\bar{y} \ldots (3).$$

From this last equation it is evident that $\bar{y}$ can never have a value greater than that given by the equation

$$2g\,(m + m' + m'' + \ldots .)\,\bar{y} = mu^2 + m'u'^2 + m''u''^2 + \ldots .$$
$$= 2g\,(m + m' + m'' + \ldots .)\,\bar{x} \text{ from } (1);$$

and therefore it is impossible for $\bar{y}$ ever to exceed $\bar{x}$ in magnitude.

It remains to consider whether, or under what circumstances, $\bar{y}$ will ever become equal to $\bar{x}$. Suppose $\bar{y} = \bar{x}$; then, from (3), we see that

$$mv^2 + m'v'^2 + m''v''^2 + \ldots . = 0,$$

and therefore, contemporaneously,

$$v = 0, \quad v' = 0, \quad v'' = 0, \ldots$$

or, from the relations (2),

$$2g\,(y-k) = u^2, \quad 2g\,(y'-k') = u'^2, \quad 2g\,(y''-k'') = u''^2 \ldots .;$$

hence we see that, if ever $\bar{y}$ become equal to $\bar{x}$, *every* molecule must have at that instant a zero velocity, or be at the highest point of one extremity of its arc of oscillation. Suppose now that t, τ, denote respectively, the time which m takes to move through the first and second portion of its arc of oscillation, the two portions being supposed to be separated

by the position of m at the instant of its detachment. Let (t', τ'), (t'', τ''), (t''', τ''') represent like times in relation to the molecules m', m'', m'''.... Then, since v, v', v''.... must be contemporaneously equal to zero in order that $\bar{y}$ may be equal to $\bar{x}$, the condition for such equality will evidently be expressed by a class of relations of the form

$$t + n(t + \tau) = t' + n'(t' + \tau') = t'' + n''(t'' + \tau'') = \ldots . (4),$$

where n, n', n''.... are all positive integers. The possibility of satisfying these relations by any finite values of n, n', n''.... as will be evident on a little reflection, will essentially depend upon the forms of the paths described by the molecules subsequently to their disconnection. In case the molecules were to commence their motions at the moment of detachment along cycloidal arcs, starting at the lowest point of each, we should have

$$t = \tau = t' = \tau' = t'' = \tau'' = \ldots .$$

and the relations (4) would become

$$2n + 1 = 2n' + 1 = 2n'' + 1 = \ldots .$$

and therefore

$$n = n' = n'' = \ldots . ;$$

or the centre of gravity of the disconnected molecules would rise at each semi-oscillation of each separate molecule into a position on a level with (G). For the sake of another illustration, suppose that the molecules are perfectly elastic and keep rebounding above a horizontal plane in vertical lines; in this case we must put τ, τ', τ''....all equal to zero, and the relations will become

$$(n + 1)\,t = (n' + 1)\,t' = (n'' + 1)\,t'' = \ldots .$$

or, since

$$u = gt, \quad u' = gt', \quad u'' = gt'' \ldots .$$

$$(n + 1)\,u = (n' + 1)\,u' = (n'' + 1)\,u'' = \ldots . ;$$

if, then, u, u', u''.... be quantities relatively commensurable, it is plain that these relations may be satisfied by admissible values of n, n', n''.... ; on the other hand, if the requisite commensurability do not subsist among the projectile velocities, it is equally evident that the relations cannot be satisfied by any finite values of n, n', n''.... ; in the one case, then, the centre of gravity of the disconnected molecules will from time to time, at finite intervals, arrive at its original altitude, while in the other case it will never attain to it in any finite time. It is needless to multiply examples of the impossibility of satisfying the formulæ (4) by any finite values of n, n', n''... it being obviously rather a singular coincidence that the requisite commensurability should subsist without previous arrangement than a state of things of probable occurrence.

We have been supposing in the preceding observations that the disconnected molecules oscillate permanently in invariable curves, and that they are free from a liability to mutual collision. We will now suppose the molecules, considered as perfectly elastic, to be capable of assuming any irregular motions whatever, and of impinging against each other. Under this more general consideration of the possible movements, the formula (2) will be no longer applicable, the values of $v, v', v'' \dots$ obviously depending not only upon $u, u', u'' \dots$ and the altitudes of ascent, but likewise upon the effects of collision. The formula (3) however, as we know by the application of the Principle of the Conservation of Vis Viva, still continues to hold good; we may therefore shew, as before, that it will be impossible for the centre of gravity of the molecules to rise to its original altitude, unless the velocities $v, v', v'' \dots$ become simultaneously equal to zero. Let t_1 denote the interval from the instant of detachment to the arrival of the molecule m at its first position of zero velocity; $t_2, t_3, t_4 \dots$ representing the intervals between its successive arrivals at such positions: let $t_1', t_2', t_3' \dots$ denote similar intervals in respect to the molecule m', and so on for the rest. Then, in order that the centre of gravity may attain to its original altitude, we must have

$$t_1 + t_2 + t_3 + \dots t_n = t_1' + t_2' + t_3' + \dots + t'_{n'} = t_1'' + t_2'' + t_3'' + \dots t''_{n''} = \dots (5),$$

the suffixes $n, n', n'' \dots$ denoting certain integral numbers.

The probability that for any finite values of $n, n', n'' \dots$ these indefinitely numerous relations should be verified, is evidently indefinitely small. We may therefore lay it down as a general practical principle, that if a mass of matter, either in its descent from rest or at the instant of its starting, be dismembered into its constituent molecules, which subsequently pursue paths prescribed by the accidental circumstances of gravity or collision, even supposing the molecules to be perfectly elastic and discarding all consideration of friction or atmospheric resistance, its centre of gravity will never again rise to a level with its original position.

The truth of this practical principle, it will be evident, depends not upon any physical impossibility of the arrival of the centre of gravity at its original altitude, but upon the infinite mathematical improbability of the conspiracy of the necessary circumstances within any finite time. The observations which we have made above, in relation to the movements of the disconnected molecules of descending bodies, will

evidently be equally applicable to the case of fluids flowing down from a higher into a lower level, the centre of gravity of which will accordingly oscillate in a lower basin without ever reaching its original altitude.

If instead of supposing the mass to be dismembered into its constituent molecules, we imagine it to be broken merely into fragments of finite magnitude, since the equation (3) will still hold good, we shall still have relations such as those of (5) for the conditions of the rise of the centre of gravity to its original height; it is important to remark, however, that the number of these relations will, according to the new hypothesis, be finite; for if three molecules of each fragment, which are not in a single straight line, ever assume zero velocities, the whole of each fragment will be in a position of instantaneous rest. Thus, if there be p fragments, there will be only $3p-1$ relations of condition; if the mass be supposed never to have fallen to pieces at all, then, according to the same reasoning, we see that there will be only two relations of condition. Thus we come to this general conclusion, that if a mass of matter in descending from any position fall to pieces, the improbability of its centre of gravity ever rising to its original altitude will be the greater, the greater be the number of the fragments; and that, when the fragments degenerate into molecules, the improbability will amount to a practical impossibility. It may perhaps not be superfluous to observe that, if all the particles of a descending body be constrained to move in parallel planes, the whole body will be at rest if any two molecules, not in the same horizontal line, be at rest, and therefore the equations of condition will be reduced to one. If the body oscillate about a fixed axis as a complex pendulum, then if any one molecule be at rest the whole body will be for a moment without motion; and therefore there will be no relations of condition, or the centre of gravity will undoubtedly rise at the end of each oscillation to its original altitude.

VII.—ON THE INVERSE CALCULUS OF DEFINITE INTEGRALS.

By George Boole.

This paper must be considered as the continuation of a paper on the *Transformation of Definite Integrals*, (*Journal*, No. XVII. p. 216.)

M. Liouville has given the following theorem as the basis of an inverse calculus of definite integrals, viz.

$$\int_0^\infty dx\, x^{n-1}\phi(x+a) = (-)^n\,\Gamma(n)\left(\frac{d}{da}\right)^{-n}\phi(a)\ldots.(28),$$

a demonstration of which will be found in an able article in this *Journal*, vol. I. p. 113. It is there observed, that if the form of $\phi(x)$ be such as to render the first side infinite, the second may be made to agree with it by the aid of the complementary function, so that the theorem is true for all forms of $\phi(x)$. M. Liouville, I believe, considers his result as only true under a limitation, and this appears to me to be the correct view. There are at least two distinct cases of the theorem, each of which admits of a convenient transformation in the case of n being fractional.

Let us consider the integral $\int dx\, x^{n-1}\phi(x+a)$, assuming in the first instance that $\phi(x)$ may be developed in ascending powers of x, and that the first exponent exceeds -1.

Put $x = -az$, and let $a = \varepsilon^{\theta}$, then

$$\begin{aligned}\int dx\, x^{n-1}\phi(a+x) &= (-)^n a^n \int dz\, z^{n-1}\phi\{a(1-z)\} \\ &= (-)^n \varepsilon^{n\theta}\int dz\, z^{n-1}\phi\{\varepsilon^{\theta}(1-z)\} \\ &= (-)^n \varepsilon^{n\theta}\int dz\, z^{n-1}(1-z)^{\frac{d}{d\theta}}\phi(\varepsilon^{\theta}).\end{aligned}$$

If we take for the limits of z, 0 and 1, the corresponding limits of x will be 0 and $-a$, and the above gives

$$\int_0^{-a} dx\, x^{n-1}\phi(a+x) = (-)^n\varepsilon^{n\theta}\frac{\Gamma(n)\,\Gamma\left(\dfrac{d}{d\theta}+1\right)}{\Gamma\left(\dfrac{d}{d\theta}+n+1\right)}\phi(\varepsilon^{\theta})\ldots.(29),$$

a result unquestionably true for all positive values of n. Let n be an integer, then

$$\begin{aligned}&\int_0^{-a} dx\, x^{n-1}\phi(a+x) \\ &\quad= (-)^n\varepsilon^{n\theta}\,\Gamma(n)\left\{\left(\frac{d}{d\theta}+n\right)\left(\frac{d}{d\theta}+n-1\right)..\left(\frac{d}{d\theta}+1\right)\right\}^{-1}\phi(\varepsilon^{\theta}) \\ &\quad= (-)^n\,\Gamma(n)\,\varepsilon^{n\theta}\varepsilon^{-n\theta}\left\{\frac{d}{d\theta}..\left(\frac{d}{d\theta}-n+1\right)\right\}^{-1}\varepsilon^{n\theta}\phi(\varepsilon^{\theta}) \\ &\quad= (-)^n\,\Gamma(n)\left\{a^n\left(\frac{d}{da}\right)^n\right\}^{-1}a^n\phi(a) \\ &\quad= (-)^n\,\Gamma(n)\left(\frac{d}{da}\right)^{-n}\phi(a)\,..\,(30),\end{aligned}$$

a conclusion easily verified by integration, it being observed that the limits of the integrations relative to a are 0 and a. We might obviously regard this theorem as true for fractional values of n; but, without thus extending our assumptions, we may see from the above, that M. Liouville's formula does

not apply when $\phi(x)$ is only susceptible of development in ascending positive powers of x.

Let us now suppose the development of $\phi(x)$ a descending one, and the exponents negative.

Put $x = az$, $a = \varepsilon^{\theta}$, as before,

$$\int dx\, x^{n-1}\phi(a+x) = \varepsilon^{n\theta}\int dz\, z^{n-1}(1+z)^{\frac{d}{d\theta}}\phi(\varepsilon^{\theta}) \ldots . (31),$$

wherein $\dfrac{d}{d\theta}$ is of *negative* interpretation. Now, by a well-known theorem, (Gregory's *Examples*, p. 473),

$$\int_0^{\infty} dz\, z^{n-1}(1+z)^{-r} = \frac{\Gamma(n)\,\Gamma(r-n)}{\Gamma(r)} \ldots\ldots (32),$$

provided that n be positive and less than $\dot{r}$. Hence, comparing with the second member of (31) taken between the same limits,

$$\varepsilon^{n\theta}\int_0^{\infty} dz\, z^{n-1}(1+z)^{\frac{d}{d\theta}}\phi(\varepsilon^{\theta}) = \varepsilon^{n\theta}\frac{\Gamma(n)\,\Gamma\left(-\dfrac{d}{d\theta}-n\right)}{\Gamma\left(-\dfrac{d}{d\theta}\right)},$$

whence, observing that the limits of x are also 0 and ∞,

$$\int_0^{\infty} dx\, x^{n-1}\phi(a+x) = \varepsilon^{n\theta}\frac{\Gamma(n)\,\Gamma\left(-\dfrac{d}{d\theta}-n\right)}{\Gamma\left(-\dfrac{d}{d\theta}\right)}\phi(\varepsilon^{\theta}) \ldots (33)$$

$$= \varepsilon^{n\theta}\,\Gamma(n)\left\{\left(-\frac{d}{d\theta}-1\right)..\left(-\frac{d}{d\theta}-n\right)\right\}^{-1}\phi(\varepsilon^{\theta})$$

$$= (-)^n\,\Gamma(n)\,\varepsilon^{n\theta}\left\{\left(\frac{d}{d\theta}+n\right)..\left(\frac{d}{d\theta}+1\right)\right\}^{-1}\phi(\varepsilon^{\theta})$$

$$= (-)^n\,\Gamma(n)\left(\frac{d}{da}\right)^{-n}\phi(a) \ldots\ldots\ldots\ldots\ldots\ldots (34).$$

Hence, it appears that M. Liouville's formula applies when $\phi(x)$ involves only negative exponents in its development, whereof the first with its sign changed exceeds n. The limits of the integrations relative to a in the second member are $\pm\infty$ and a. § 6.

Finally, let us suppose n frational.

The object of investigation being to determine the function under the sign of integration, when the definite integral is given, let us assume that within the supposed limits, $\int dx\, x^{n-1}\phi(a+x) = \psi(a)$, a known function, and that we wish

to ascertain the form of $\phi(a)$. In the case of the ascending development we have from (29), on changing a into ϵ^θ,

$$\psi(\epsilon^\theta) = (-)^n \Gamma(n)\, \epsilon^{n\theta} \frac{\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}+n+1\right)} \phi(\epsilon^\theta),$$

$$\phi(\epsilon^\theta) = (-)^n \frac{1}{\Gamma(n)} \frac{\Gamma\left(\frac{d}{d\theta}+n+1\right)}{\Gamma\left(\frac{d}{d\theta}+1\right)} \epsilon^{-n\theta} \psi(\epsilon^\theta) \ldots\ldots (35)$$

$$= (-)^n \frac{1}{\Gamma(n)} \epsilon^{-n\theta} \frac{\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}-n+1\right)} \psi(\epsilon^\theta) \ldots\ldots\ldots (36).$$

Let i be the integer next below n, being 0 when n is a proper fraction,

$$\epsilon^{-n\theta} \frac{\Gamma\left(\frac{d}{d\theta}+1\right)}{\Gamma\left(\frac{d}{d\theta}-n+1\right)} \psi(\epsilon^\theta) = \epsilon^{-n\theta} \frac{d}{d\theta}\left(\frac{d}{d\theta}-1\right)\ldots\left(\frac{d}{d\theta}-i\right) \frac{\Gamma\left(\frac{d}{d\theta}-i\right)}{\Gamma\left(\frac{d}{d\theta}-n+1\right)} \psi(\epsilon^\theta)$$

$$= a^{-n}\left\{a^{i+1}\left(\frac{d}{da}\right)^{i+1}\right\} \frac{1}{\Gamma(i-n+1)} \frac{\Gamma\left(\frac{d}{d\theta}-i\right)\Gamma(i-n+1)}{\Gamma\left(\frac{d}{d\theta}-n+1\right)} \psi(\epsilon^\theta)$$

$$= \frac{a^{i-n+1}}{\Gamma(i-n+1)} \left(\frac{d}{da}\right)^{i+1} \int_0^1 dv\, v^{\frac{d}{d\theta}-i-1} (1-v)^{i-n} \psi(\epsilon^\theta)$$

$$= \frac{a^{i-n+1}}{\Gamma(i-n+1)} \left(\frac{d}{da}\right)^{i+1} \int_0^1 dv\, v^{-i-1} (1-v)^{i-n} \psi(av),$$

$$\therefore \phi(a) = (-)^n \frac{a^{i-n+1}}{\Gamma(n)\Gamma(i-n+1)} \left(\frac{d}{da}\right)^{i+1} \int_0^1 dv\, v^{-i-1}(1-v)^{i-n}\psi(av) \ldots (37).$$

Similarly for the descending development of $\phi(x)$, and for negative exponents, we have, by (33),

$$\psi(\epsilon^\theta) = \epsilon^{n\theta} \frac{\Gamma(n)\,\Gamma\left(-\frac{d}{d\theta}-n\right)}{\Gamma\left(-\frac{d}{d\theta}\right)} \phi(\epsilon^\theta)$$

$$\phi(\epsilon^\theta) = \frac{1}{\Gamma(n)} \frac{\Gamma\left(-\dfrac{d}{d\theta}\right)}{\Gamma\left(-\dfrac{d}{d\theta} - n\right)} \epsilon^{-n\theta} \psi(\epsilon^\theta) \ldots\ldots (38)$$

$$= \frac{1}{\Gamma(n)} \epsilon^{-n\theta} \frac{\Gamma\left(-\dfrac{d}{d\theta} + n\right)}{\Gamma\left(-\dfrac{d}{d\theta}\right)} \psi(\epsilon^\theta).$$

Now let i be the integer next below n, then

$$\Gamma\left(-\frac{d}{d\theta} + n\right) = \left(-\frac{d}{d\theta} + n - 1\right) \ldots \left(-\frac{d}{d\theta} + n - i\right) \Gamma\left(-\frac{d}{d\theta} + n - i\right)$$

$$\Gamma\left(-\frac{d}{d\theta}\right) = \left(-\frac{d}{d\theta}\right)^{-1} \Gamma\left(-\frac{d}{d\theta} + 1\right),$$

$$\therefore\ \phi(\epsilon^\theta)$$

$$= \frac{1}{\Gamma(n)} \epsilon^{-n\theta} \left(-\frac{d}{d\theta} + n - 1\right) \ldots \left(-\frac{d}{d\theta} + n - i\right)\left(-\frac{d}{d\theta}\right) \frac{\Gamma\left(-\dfrac{d}{d\theta} + n - i\right)}{\Gamma\left(-\dfrac{d}{d\theta} + 1\right)}$$

$$= (-)^{i+1} \frac{1}{\Gamma(n)} \epsilon^{-n\theta} \left(\frac{d}{d\theta} - n + i\right) \ldots \left(\frac{d}{d\theta} - n + 1\right) \frac{d}{d\theta} \frac{\Gamma\left(-\dfrac{d}{d\theta} + n - i\right)}{\Gamma\left(-\dfrac{d}{d\theta} + 1\right)} \psi(\epsilon^\theta)$$

$$= (-)^{i+1} \frac{1}{\Gamma(n)} \epsilon^{-n\theta} \epsilon^{(n-i)\theta} \frac{d}{d\theta}\left(\frac{d}{d\theta} - 1\right) \ldots \left(\frac{d}{d\theta} - i + 1\right) \epsilon^{(i-n)\theta} \frac{d}{d\theta} \frac{\Gamma\left(-\dfrac{d}{d\theta} + n - i\right)}{\Gamma\left(-\dfrac{d}{d\theta} + 1\right)} \psi($$

$$= (-)^{i+1} \frac{1}{\Gamma(n)} \left(\frac{d}{da}\right)^i a^{i-n+1} \frac{d}{da} \frac{1}{\Gamma(i - n + 1)} \frac{\Gamma\left(-\dfrac{d}{d\theta} + n - i\right)\Gamma(i - n + 1)}{\Gamma\left(-\dfrac{d}{d\theta} + 1\right)} \psi$$

$$= (-)^{i+1} \frac{1}{\Gamma(n)\,\Gamma(i - n + 1)} \left(\frac{d}{da}\right) a^{i-n+1} \frac{d}{da} \int_0^1 dv\, v^{n-i-1} (1 - v)^{i-n} \psi\left(\frac{a}{v}\right) \ldots (3$$

$$\text{since}\quad v^{-\frac{d}{d\theta}} \psi(\epsilon^\theta) = \psi\left(\frac{a}{v}\right).$$

We shall somewhat simplify the above formula by changing i into $i - 1$, and in (30) and (37), x into $-x$, in order to avoid the negative limit. Our results will then be comprised in the following theorems.

1^{st}. If, n being a positive fraction, $\int_0^a dx\, x^{n-1}\phi(a-x) = \psi(a)$, and if i being the integer next above n, $i-1$ be less than the first exponent in the ascending development of $\psi(a)$, then

$$\phi(a) = \frac{a^{i-n}}{\Gamma(n)\Gamma(i-n)} \left(\frac{d}{da}\right)^i \int_0^1 dv\, v^{-i}(1-v)^{i-n-1}\psi(av);$$

but if n be a positive integer less than the same first exponent,

$$\phi(a) = \frac{1}{\Gamma(n)} \left(\frac{d}{da}\right)^n \psi(a).$$

2^{nd}. If, n being a positive fraction, $\int_0^\infty dx\, x^{n-1}\phi(a+x) = \psi(a)$, the exponents in the development of $\psi(a)$ being all negative, and if i be the integer next above n, then

$$\phi(a) = (-)^i \frac{1}{\Gamma(n)\Gamma(i-n)} \left(\frac{d}{da}\right)^{i-1} a^{i-n} \frac{d}{da}\int_0^1 dv\, v^{n-i}(1-v)^{i-n-1}\psi\left(\frac{a}{v}\right);$$

but if n be a positive integer,

$$\phi(a) = (-)^n \frac{1}{\Gamma(n)} \left(\frac{d}{da}\right)^n \psi(a).$$

The application of these formulæ, as it presents no difficulties, so it requires no illustration. They enable us, whenever $\psi(a)$ satisfies the necessary conditions, to determine $\phi(a)$ either in the form of a definite integral, or in that of a series; but for the latter object it is perhaps simpler, changing x into ε^θ, to employ at once the untransformed equations (35) and (38).

I apprehend that the same principles may be extended, by aid of the theorem of M. Dirichlet, to the inverse calculus of definite multiple integrals. At present, however, I have not leisure to pursue this inquiry, nor am I so much concerned to multiply results as to establish principles.

Lincoln, Oct. 26, 1842.

VIII.—ON A NEW SPECIES OF EQUATIONS OF DIFFERENCES.

THE sort of equation to which the heading refers is that in which the degree of the equation with respect to one variable is defined by the other, as in $u_{x,y} = u_{x-y,\,x+y}$, where $u_{x,y}$ is a function of the integers x and y. A problem of some interest leads to such an equation, and will tend to shew the sort of solution which may be given.

It is well known that the number of ways in which the number n may be put together is 2^{n-1}: thus 4 can be con-

structed in 2^3 ways; namely, 4, 31, 13, 22, 211, 121, 112, 1111. But in this theorem orders are counted; thus, 112 is considered as different from 121, and both from 211. The corresponding problem, namely, to find in how many ways the number n can be constructed, different orders not being considered as different ways, is of much greater difficulty, and I am not aware of any thing having been written upon it.

In the first place, if k be not less than n, the number of ways in which n can be put together is the same as that in which $n+k$ can be composed of numbers one of which is k: thus 16 can be $8+\ldots.$ in as many ways as 8 can be put together. If then $u_{x,y}$ represent the number of ways in which x can be made up of y, and numbers not exceeding y, the problem is solved when $u_{x,y}$ is found: for the number of ways in which x can be put together is $u_{2x+k,\,x+k}$, whatever may be the value of k, from 0 inclusive.

Again, it is obvious that the number of ways in which any number, as 16, can be composed of numbers no higher than 6 and not all lower, or $u_{16.6}$, is the sum of the ways of composing 10 out of numbers not exceeding 1, 2...6 Every way in which 10 can be so formed, gives 16 by adding 6. Hence

$$u_{x,y} = u_{x-y,1} + u_{x-y,2} + \ldots + u_{x-y,y},$$

$$u_{x+1,y+1} = u_{x-y,1} + u_{x-y,2} + \ldots + u_{x-y,y+1}.$$

Hence $$u_{x+1,y+1} - u_{x,y} = u_{x-y,y+1},$$

or $$u_{x,y} - u_{x-1,y-1} = u_{x-y,y},$$

an equation of the y^{th} degree with respect to x.

This equation gives a tolerably easy mode of constructing a table: for all numbers up to 10 it is as follows; $u_{x,1}$, $u_{x,x}$, and $u_{x,x-1}$ being always units.

	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	1	1							
4	1	2	1	1						
5	1	2	2	1	1					
6	1	3	3	2	1	1				
7	1	3	4	3	2	1	1			
8	1	4	5	4	3	2	1	1		
9	1	4	7	6	5	3	2	1	1	
10	1	5	8	9	7	5	3	2	1	1

The rule being, after the units are written down, from the y^{th} number of the x^{th} line by adding the $(y-1)^{\text{th}}$ number of the $(x-1)^{\text{th}}$ line to the $(x-y)^{\text{th}}$ number in the x^{th} line. Thus 10 can be formed out of numbers one of which is 4, and no other exceeds 4, in 9 ways: these 9 ways are 442, 4411, 433, 4321, 43111, 4222, 42211, 421111, 4111111.

As soon as we can express $u_{x-1,\,y-1}$ in terms of x and y, we have then an equation of the y^{th} order for determining $u_{x,\,y}$. Say that $u_{x-1,\,y-1} = V_{x,\,y}$, we have then,

$$u_{x,\,y} - u_{x-y,\,y} = V_{x,\,y},$$

which, considering y as a constant, is of the form

$$w_x - w_{x-y} = V_x.$$

From this consideration we can arrive at the general form of the solution, and this is all I shall here attempt. First we have $u_{x,\,1} = 1$, so that

$$u_{x,\,2} - u_{x-2,\,2} = 1;$$

the complete integral of which, the constants being found from $u_{1,\,2} = 0$, $u_{2,\,2} = 1$, is

$$u_{x,\,2} = \frac{x}{2} - \frac{1}{4} + \frac{1}{4}(-1)^x.$$

Therefore $$u_{x,\,3} - u_{x-3,\,3} = \frac{x-1}{2} - \frac{1}{4} + \frac{1}{4}(-1)^{x-1};$$

the complete integral of which, determining the constants from $u_{1,\,3} = 0$, $u_{2,\,3} = 0$, $u_{3,\,3} = 1$, is

$$u_{x,\,3} = \frac{6x^2 - 7 - 9(-1)^x + 8(\beta^x + \gamma^x)}{72},$$

where β and γ are the imaginary cube roots of unity. This may be expressed as follows:—The number of ways in which x can be composed of numbers, none greater and not all less than 3, is

$$\frac{x^2}{12},\ \frac{x^2-1}{12},\ \frac{x^2-4}{12},\ \frac{x^2+3}{12},\ \frac{x^2-4}{12},\ \text{or}\ \frac{x^2-1}{12},$$

according as x divided by 6 leaves a remainder 0, 1, 2, 3, 4, or 5. If we were now to attempt to generalise, we might suppose that

$$u_{x,\,y} \text{ is of the form } A_{y-1} + P_2 + \ldots + P_y,$$

where A_n represents a rational and integral function of the n^{th} degree, and P_n is a periodic or circulating function of the n^{th} order, or going through a cycle of n values. But this

inference would not be quite correct. The terms of each successive integration are of the form

$$\text{definite constant} \times \mu^x \Sigma(\mu^{-x} V_x), \quad (\mu^n = 1).$$

When V_x is a periodic term of the form $C\beta^x$, β being some previous root of unity (say the m^{th}), generally speaking $\mu^x\Sigma.\mu^{-x}V_x$ is a simple circulating function of the m^{th} kind; and if V_x be of the form $A_n\beta^x$, the form resulting from integration is also $A_n\beta^x$. But if $\beta = \mu$, which happens among the cases arising when n is a multiple of m, then $\mu^x\Sigma.\mu^{-x}V_x$ takes the form $\mu^x A_{n+1}$, or the function of x is raised a unit in dimension. Hence the most general form of the solution of $u_{x,y} - u_{x-1,y-1} = u_{x-y,y}$, is

$$u_{x,y} = A_{y-1} + A_{a_2}P_2 + A_{a_3}P_3 + \ldots . A_{a_y}P_y,$$

where a_n, the dimension of the multiplier of P_n, is the integer in $n - y$ divided by y.

The next step gives for $u_{x,4}$ the denominator 864, and the following numerator:

$$6x^3 + 18x^2 - 27x - 39 + (27x + 27)(-1)^x$$
$$+ 32(\beta^{x-1} + \gamma^{x-1} - \beta^x - \gamma^x) + 54\{(\sqrt{-1})^x + (-\sqrt{-1})^x\},$$

this is the 144^{th} part of

(0) $x^3 + 3x^2$,	(6) $x^3 + 3x^2 - 36$,
(1) $x^3 + 3x^2 - 9x + 5$,	(7) $x^3 + 3x^2 - 9x + 5$,
(2) $x^3 + 3x^2 - 20$,	(8) $x^3 + 3x^2 + 96$,
(3) $x^3 + 3x^2 - 9x - 27$,	(9) $x^3 + 3x^2 - 9x - 27$,
(4) $x^3 + 3x^2 + 32$,	(10) $x^3 + 3x^2 - 4$,
(5) $x^3 + 3x^2 - 9x - 11$,	or (11) $x^3 + 3x^2 - 9x - 11$.

according as x divided by 12 gives the remainder 0, 1, 2. . . . or 11.

It thus appears that $u_{x,3}$ may be described as the integer nearest to $x^2 \div 12$, and $u_{x,4}$ as the integer nearest to $(x^3 + 3x^2) \div 144$ or $(x^3 + 3x^2 - 9x) \div 144$, according as x is even or odd. Probably this simple species of description might be continued.

IX.—NOTES ON MAGNETISM. NO. I.

By R. L. Ellis, M.A. Fellow of Trinity College.

A geometrical construction, by means of which the action of a small magnet on a distant particle of free magnetism may be readily determined, is mentioned in a memoir by Weber, on the Bifilar Magnetometer (*Scientific Memoirs*, II. p. 270). It is due to Gauss, but I do not know where he has demon-

strated it. A proof of it may be acceptable to some readers of the *Journal*.

I begin by enunciating the construction in question, which will be easily understood without a figure. Let AB be a small bar magnet, c its centre, P the particle of magnetism on which it acts. Join cP, draw PD perpendicular to it, meeting cP produced in D. Let $cQ = \frac{1}{3} cD$. Join PQ. Then PQ or QP (according to the sign of the magnetism of P and the direction of the poles of AB) is the direction of the action of AB on P; and $\frac{Mm}{cP^3} \frac{PQ}{cQ}$ is its magnitude, M being the measure of the magnetism of AB, m that of the magnetism of P.

The dimensions of the magnet being small, and its length in the direction of its axis being much greater than its breadth or thickness, we may proceed as follows.

Conceive the magnet to be composed of a series of intense particles ranged along its axis. Let s be the distance of any one of them from c, μds the measure of its magnetism. Also let $cP = r$, and call the angle cP makes with cD, θ.

Then the distance of the particle in question from P, is $(r^2 + s^2 - 2rs \cos \theta)^{\frac{1}{2}}$, and consequently its action on P is

$$\frac{m\mu ds}{r^2 + s^2 - 2rs \cos \theta},$$

magnetic attraction being supposed to follow the ordinary law. The component of this action along cD, is

$$\frac{m\mu ds}{(r^2 + s^2 - 2rs \cos \theta)^{\frac{3}{2}}} (r \cos \theta - s).$$

This is approximately equal to

$$\frac{m\mu}{r^3} (1 + 3 \frac{s}{r} \cos \theta) (r \cos \theta - s) \, ds,$$

or to

$$\frac{m\mu}{r^3} \{r \cos \theta + (3 \cos^2 \theta - 1) s\} \, ds;$$

and therefore the total action of AB on P, parallel to cD, is

$$\frac{m \cos \theta}{r^2} \int \mu ds + m \frac{3 \cos^2 \theta - 1}{r^3} \int \mu s ds,$$

the integrals being taken along the whole length of the magnet. Consequently $\int \mu ds = 0$,

since the aggregate magnetism of a magnet, each element being taken with its proper sign, is zero. Again, $\int \mu s ds$

being the *moment* of the magnetism of AB, is the measure of its magnetic power, or what we have called M.

Consequently the action parallel to cD is

$$\frac{Mm}{r^3}(3\cos^2\theta - 1) \ldots\ldots\ldots\ldots (1).$$

The action of the element at s, perpendicular to cD, is

$$\frac{m\mu ds}{(r^2 + s^2 - 2rs\cos\theta)^{\frac{3}{2}}} r\sin\theta,$$

or, approximately,

$$\frac{m\mu}{r^2}(1 + 3\frac{s}{r}\cos\theta)\sin\theta ds.$$

The total action perpendicular to cD is, therefore,

$$\frac{3Mm}{r^3}\sin\theta\cos\theta \ldots\ldots\ldots\ldots (2).$$

The equation to the resultant of these two forces is, since the line passes through P,

$$\frac{y - r\sin\theta}{3\sin\theta\cos\theta} = \frac{x - r\cos\theta}{3\cos^2\theta - 1}.$$

For $y = 0$, or at the point Q,

$$x_0 = r\cos\theta - \tfrac{1}{3}r\sec\theta(3\cos^2\theta - 1),$$

$$\text{or} \quad x_0 = \tfrac{1}{3}r\sec\theta.$$

Now $cD = r\sec\theta$, therefore

$$cQ = \tfrac{1}{3}cD,$$

which proves the first part of the construction.

Next, to find the magnitude of the whole action, square and add (1) and (2), then

$$\frac{Mm}{r^3}(1 - 6\cos^2\theta + 9\cos^4\theta + 9\sin^2\theta\cos^2\theta)^{\frac{1}{2}},$$

$$\text{or} \quad \frac{Mm}{r^3}(1 + 3\cos^2\theta)^{\frac{1}{2}}$$

is the magnitude sought. Now

$$\begin{aligned} PQ &= r\left\{\sin^2\theta + (\cos\theta - \tfrac{1}{3}\sec\theta)^2\right\}^{\frac{1}{2}} \\ &= r(1 - \tfrac{2}{3} + \tfrac{1}{9}\sec^2\theta)^{\frac{1}{2}} \\ &= \tfrac{1}{3}r(\sec^2\theta + 3)^{\frac{1}{2}}, \end{aligned}$$

and $\qquad cQ = \tfrac{1}{3}r\sec\theta.$

Therefore $\qquad \dfrac{PQ}{cQ} = (1 + 3\cos^2\theta)^{\frac{1}{2}}.$

Consequently, if R be the magnitude of the resultant sought,

$$R = \frac{Mm}{cP^3}\,\frac{PQ}{cP},$$

which was to be proved.

The preceding formulæ enable us to determine all the circumstances of the mutual action of two magnets, which are such as to fulfil the conditions of our hypothesis.

For instance, in the memoir *Intensitas vis Magneticæ Terrestris,* Gauss has shown that if a magnet and a needle be placed at right angles to one another, then the moment of rotation of the needle due to the action of the magnet is approximately twice as great when the line of the axis of the magnet passes through the centre of the needle, as it is when the line of the axis of the needle passes through the centre of the magnet.

In neither case has the transverse force (2) any tendency to produce rotation. Its effect is destroyed by the fixed centre of the needle.

Let y be the distance of any element of the needle from the centre, mdy its magnetism. Then, in the first case, we shall have $\theta = \frac{y}{R}$ approximately, (R being the distance between the centre of the magnet and that of the needle). Consequently the force (1) may be expressed by the formula

$$\frac{Mmdy}{(R^2 + y^2)^{\frac{3}{2}}}\left(2 - 3\,\frac{y^2}{R^2}\right) \text{ since } \cos^2\theta = 1 - \theta^2 \text{ nearly;}$$

which, neglecting y^2, becomes

$$\frac{2Mmdy}{R^3};$$

the moment of this round the centre of the needle is $\frac{2Mmydy}{R^3}$, and the total moment is, therefore,

$$\frac{2MM'}{R^3} \quad \text{where } M' = \int mydy.$$

In the second case, $\theta = 0$ for every element of the needle, and the moment sought is, therefore,

$$\frac{MM'}{R^3}, \text{ since (1) then becomes } -\frac{Mm}{r^3},$$

and the sign is immaterial. The moment in this case is, therefore, one half of what it was before, which was to be proved.

This result is included in the following general investigation, in which we shall ascertain the moment of rotation due

to the action of one magnet or needle upon another, whatever be their relative positions.

The data by which we shall suppose the position of the magnets to be determined, are the distance between their centres, the angles which the axis of each respectively makes with the line joining their centres, and the angle between the axes themselves. The last element may be readily replaced by the dihedral angle between two planes which intersect in the line joining the centres of the magnets, and one of which passes through the axis of one of the magnets while the other passes through that of the other.

Let the axis of the magnet, whose action on the other is to be calculated, be taken as axis of x, the centre of the magnet being the origin of co-ordinates. Let x, y, z be the co-ordinates of any point in the other magnet. Then, if $r^2=x^2+y^2+z^2$,

$$3\cos^2\theta - 1 = \frac{3x^2 - r^2}{r^2}, \quad \text{and} \quad \sin\theta\cos\theta = \frac{x\sqrt{(y^2+z^2)}}{r^2}.$$

Again, let a, b, c be the co-ordinates of the centre of the second magnet, α, β, γ the cos angles its axis makes with the co-ordinate axes, ρ the distance of any element $d\rho$ from the centre, X, Y, Z the forces on $d\rho$ parallel to the axes of co-ordinates, m the intensity of $d\rho$. Then

$$X = \frac{Mm}{r^5}(3x^2 - r^2)\,d\rho,$$

$$Y = \frac{3Mm}{r^5}\,xyd\rho,$$

$$Z = \frac{3Mm}{r^5}\,xzd\rho.$$

Let G, H, K be the moments of these forces about lines drawn through the centre of the second magnet parallel to the axes of co-ordinates,

$$G = \frac{3Mm}{r^5}\,x\,\{y(z-c) - z(y-b)\}\,d\rho,$$

$$H = \frac{Mm}{r^5}\,\{3xz\,(x-a) - (3x^2 - r^2)(z-c)\}\,d\rho,$$

$$K = \frac{Mm}{r^5}\,\{(3x^2 - r^2)(y-b) - 3xy\,(x-a)\}\,d\rho.$$

Now $\qquad x = a + \alpha\rho, \quad y = b + \beta\rho, \quad z = c + \gamma\rho.$

Hence, neglecting the square, &c. of ρ,

$$G = \frac{3Mm}{R^5}\,a\,(b\gamma - c\beta)\,\rho d\rho,$$

$$H = \frac{Mm}{R^5}\left\{3aca - (3a^2 - R^2)\gamma\right\}\rho d\rho,$$

$$K = \frac{Mm}{R^5}\left\{(3a^2 - R^2)\beta - 3aba\right\}\rho d\rho,$$

$$\text{where} \quad R^2 = a^2 + b^2 + c^2.$$

Integrating for ρ, we find, for the total moments,

$$(G) = \frac{3MM'}{R^5}\, a\,(b\gamma - c\beta),$$

$$(H) = \frac{3MM'}{R^5}\, a\,(ca - a\gamma) + \frac{MM'}{R^3}\,\gamma,$$

$$(K) = \frac{3MM'}{R^5}\, a\,(a\beta - ba) - \frac{MM'}{R^3}\,\beta,$$

M' being for the second magnet what M is for the first.

Let L be the resultant of these three moments, then

$$L^2 = (G)^2 + (H)^2 + (K)^2.$$

Consequently, since $a^2 + \beta^2 + \gamma^2 = 1$, we shall have

$$\begin{aligned} L^2 = {} & \left(\frac{3MM'}{R^5}\right)^2 a^2\left\{R^2 - (aa + b\beta + c\gamma)^2\right\} \\ & + 6\left(\frac{MM'}{R^4}\right)^2\left\{aa\,(aa + b\beta + c\gamma) - a^2\right\} \\ & + \left(\frac{MM'}{R^3}\right)^2(\beta^2 + \gamma^2). \end{aligned}$$

Now, let θ, θ' be the angles which the axes of the two magnets make respectively with the line joining their centres, and let ϕ be the angle between the axes themselves. Then

$$a = R\cos\theta \quad\text{and}\quad aa + b\beta + c\gamma = R\cos\theta', \quad\text{also}\quad a = \cos\phi.$$

Consequently

$$L^2 = \left(\frac{MM'}{R^3}\right)^2\left\{9\cos^2\theta\sin^2\theta' + 6\cos\theta\,(\cos\theta'\cos\phi - \cos\theta) + \sin^2\phi\right\},$$

or $$L = \frac{MM'}{R^3}\left\{1 + 3\cos^2\theta - (3\cos\theta\cos\theta' - \cos\phi)^2\right\}^{\frac{1}{2}}.$$

Let χ be the dihedral angle already mentioned, then

$$\cos\phi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\chi.$$

Consequently the last equation becomes

$$L = \frac{MM'}{R^3}\left\{1 + 3\cos^2\theta - (2\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\chi)^2\right\}^{\frac{1}{2}},$$

the required expression.

ADDENDUM TO ART. II.

The method of this paper may be applied to the integration of any linear equation with constant coefficients, by means of the following theorems, the first of which answers to integration by parts, the second to John Bernoulli's Theorem. Let D denote the operation of finding a differential coefficient, then

$$(D+a)^{-1}.PQ = P(D+a)^{-1}Q - (D+a)^{-1}\{P'(D+a)^{-1}Q\},$$

P' being the result of DP: and, abbreviating $D+a$ into Θ,

$$\Theta^{-1}.PQ = P\Theta^{-1}Q - P'\Theta^{-2}Q + P''\Theta^{-3}Q\ldots.$$

which, if P be rational and integral, gives $\Theta^{-1}.PQ$ in terms of $\Theta^{-1}Q$, &c.

Thus, to find all the preceding equations of

$$(D-1)^6(D-2)^8y = X,$$

those of the 13th degree, for instance: operate upon

$$(D-1)^{-1}\{x^m(D-1)^6(D-2)^8y\} = (D-1)^{-1}(x^mX),$$
$$(D-2)^{-1}\{x^n(D-2)^8(D-1)^6y\} = (D-2)^{-1}(x^nX),$$

for all integer values of m and n, from $m=0$ to $m=5$, and from $n=0$ to $n=7$. We shall thus have the 14 equations of the 13th degree; and from each of these, exactly as in the preceding part of this paper, others of the 12th, &c. degrees may be obtained.

The criterion which, being satisfied, shows that

$$\Sigma\{P_m(D+a)^my\}$$

will, when operated upon by the symbol $(D+a)^{-1}$, not show that symbol in the result, precisely resembles the corresponding criterion when $a=0$, or the case of simple integration.

The preceding method may be equally applied to finding the intermediate equations of linear partial differential equations with constant coefficients, or of equations of differences.

With respect to equations of differences, the process answering to integration by parts is as follows. Let E signify the operation of changing x into $x+\Delta x$, then, P and Q being functions of x,

$$(E+a)^{-1}.(PQ) = P(E+a)^{-1}Q - (E+a)^{-1}\{\Delta P.(E+a)^{-1}EQ\},$$

and the extension of John Bernoulli's theorem is, $E+a$ being Θ,

$$\Theta^{-1}(PQ) = P\Theta^{-1}Q - \Delta P.\Theta^{-2}EQ + \Delta^2P\Theta^{-3}E^2Q - \ldots.$$

which terminates when P is rational and integral.

In applying this to such an equation as

$$(E-a)^m(E-b)^n\ldots.y+X,$$

instead of using x^k as a multiplier, it would be more convenient to use the factorial $x^{k|1}$.

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I.—ON A LAW EXISTING IN THE SUCCESSIVE APPROXIMATIONS TO A CONTINUED FRACTION.

LET there be a set of symbols 1_1, 2_1, 3_1, 4_1, &c. from which 1_2, 2_2, 3_2, &c, 1_3, 2_3, 3_3, &c., are formed as follows, $1_2 = 2_1\, 1_1 + 1$, $1_3 = 3_1\, 1_2 + 1_1$, $1_4 = 4_1\, 1_3 + 1_2$, &c., $2_2 = 3_1\, 2_1 + 1$, $2_3 = 4_1\, 2_2 + 2_1$, &c., so that the numerators and denominators of the successive approximations to the continued fraction $\dfrac{1}{n_1+}\,\dfrac{1}{n_2+}\,\dfrac{1}{n_3+}\ldots\ldots$ may be readily obtained in the usual manner.

The law of formation is

$$y_x = (y + x - 1)_1\, y_{x-1} + y_{x-2}.$$

Another law of formation is

$$y_x = y_n\, (y + n)_{x-n} + y_{n-1}\, (y + n + 1)_{x-n-1},$$

which may be easily proved. Though this double numerical symbol is convenient in the exhibition of laws, it will be better in ordinary work to write a, b, c, d, &c. for 1, 2, 3, 4, and to drop the suffixes in a, b, c, &c.

The object of this paper is the investigation of the law of formation exhibited in a, $ab + 1$, $abc + a + c$, &c., or $1_1\, 1_2\, 1_3$ &c. in terms of 1_1, 2_1, 3_1, &c. Of these it is evident that the n^{th} is made up of terms of $n, n-2, n-4, \ldots\ldots$ dimensions, the term of no dimension being always unity. If we signify by $(abc.\ldots.k)_n$ the collection of terms of the n^{th} dimension which appear in the first step which introduces the letter k, we have for the several steps

$(a)_1$, $(ab)_2 + (ab)_0$, $(abc)_3 + (abc)_1$, $(abcd)_4 + (abcd)_2 + (abcd)_0$, and so on.

To find $(abc.\ldots.k)_n$ the rule is as follows. Take all those combinations of $a, b, \ldots.k$, in which there are no breaks ex-

cept in pairs: thus in $(abcde)_3$ acd cannot occur, for a single letter b has dropt out by itself: but cde, ade, abe, abc can and do occur, and no others. Thus the terms of the third dimension in 1_5 are $cde + ade + abe + abc$.

First, there can be no breaks but in pairs. When k is introduced for the first time, the terms then factored with k are factored with l at the next step, and no break at all is made from k. But at the next step, the terms factored with k two steps before, enter by simple addition, and are not factored with m. Consequently a term into which k entered at the n^{th} step, is never seen followed by m without l, but has nothing before n if l be missing.

Secondly, all terms in which all breaks are pairs, or multiples of pairs, actually do enter. For instance, $ehim$ must ultimately enter. It is clear from the formation that even steps must all contain unity, therefore the fifth step must contain e, which enters therefore by addition in the seventh step. Hence the eighth step has eh and the ninth ehi, which enters by addition into the eleventh step, and so the twelfth step has $ehim$, which is accordingly found in the fourteenth, sixteenth, &c. steps.

From these converse propositions it follows that, if k represent the m^{th} letter, the m^{th} step, or ${}^1m$, is

$$(ab.\dots k)_m + (ab.\dots k)_{m-2} + (ab.\dots k)_{m-4} + \dots\dots$$

Thus the sixth step, written down independently of the preceding ones, is

$$abcdef + cdef + adef + abef + abcf + abcd + cf + cd + ad + ab + af + ef + 1.$$

When there are m letters in the step, the number of terms of the $(m-2n)^{\text{th}}$ dimension is the number of ways in which $m-2n$ can be taken out of $m-n$. And generally, the number of ways in which n sets of z consecutive letters each can be abstracted from m letters, is the number of ways in which $m - zn$ (or n) can be removed out of $m-(z-1)\,n$. To prove this, let $Z_{n,m}$ represent the number of ways just mentioned; in every one of these ways, either the last letter is among those abstracted, or it is not; $Z_{n-1,m-z}$ is the number of ways in which it is abstracted, $Z_{n,m-1}$ that in which it is not. Consequently

$$Z_{n,m} = Z_{n,m-1} + Z_{n-1,m-z}.$$

Let $m = nz + k$, whence $\Delta Z_{n-1,(n-1)z+k} = Z_{n,nz+k-1}$

$$\text{or}\quad Z_{n,nz+k} = \Sigma Z_{n+1,(n+1)z+k-1} \dots\dots$$

Now $Z_{n,nz} = 1$, and $Z_{1,z+k} = k+1$; whence the successive operations by which to ascend from $Z_{n,nz}$ to $Z_{n,nz+k}$ for any value of k, are 1. Change n into $n+1$. 2. Perform the

operation of finite summation. 3. Determine the arbitrary constant so that $Z_{1,2\cdot k}$ may be $k+1$. This gives us for values, beginning from $k=0$,

$$1,\ n+1,\ \frac{(n+1)(n+2)}{2},\ \frac{(n+1)(n+2)(n+3)}{2.3},\ \&c.,$$

which contain the theorem to be proved.

It appears then that in the m^{th} approximation, the fourth dimension $(abc\ldots k)_m$ has one term only, but that $(abc\ldots k)_{m-2n}$ has

$$\frac{(m-n)(m-n-1)\ldots.(m-2n+1)}{1.2.3\ldots.n} \text{ terms.}$$

Accordingly, if $a=b=c=d$, &c., we have, for the n^{th} step,

$$a^m+(m-1)a^{m-2}+\frac{m-2}{1}\frac{m-3}{2}a^{m-4}+\frac{m-3}{1}\frac{m-4}{2}\frac{m-5}{3}a^{m-6}+\ldots.$$

It may be worth the noting, that the sum of the coefficients just written down, or the value of the expression when $a=1$, is the number of distinct ways (different orders counting as different ways) in which $m+1$ can be compounded by addition of *odd* numbers; so that, m increasing without limit, the number of ways in which m can be made of odd numbers is to the same for $m+1$ ultimately in the ratio of the segments of a line which is divided in what Euclid calls extreme and mean ratio.

A. DE M.

March 20, 1844.

II.—NOTES ON CONIC SECTIONS.

(1) THE eccentric anomaly may be made use of to prove several properties of the ellipse in a very simple manner, especially in all cases when conjugate diameters are concerned, as the following examples will shew.

There is a very simple but important theorem respecting the eccentric anomalies of the extremities of two conjugate diameters, which has not been noticed so far as I am aware. It is this:

If ϕ and ϕ' be the eccentric anomalies of P and D respectively, $\phi'-\phi=\dfrac{\pi}{2}$.

We may put the equation to the ellipse in the form of two equations, viz. $x=a\cos\phi,\quad y=b\sin\phi,$

the angle ϕ is called the eccentric anomaly of the point xy.

Now xy and $x'y'$ being the co-ordinates of P and D, and ϕ, ϕ' the corresponding eccentric anomalies, we have

$$\left.\begin{array}{ll} x = a\cos\phi, & y = b\sin\phi \\ x' = a\cos\phi', & y' = b\sin\phi' \end{array}\right\} \ldots\ldots\ldots (1).$$

And, by a well known theorem,

$$\frac{y}{x}\frac{y'}{x'} = -\frac{b^2}{a^2},$$

$$\text{or}\quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = 0;$$

substituting the values (1) in this equation, we find immediately

$$\cos(\phi' - \phi) = 0,$$

and therefore $\qquad \phi' - \phi = \dfrac{\pi}{2}$. Q. E. D.

(2) Let $r\theta$, $r'\theta'$, be the polar co-ordinates of P and D, then

$$r^2 = x^2 + y^2 = a^2\cos^2\phi + b^2\sin^2\phi,$$

but by the theorem just proved r' is obtained from r by putting $\phi + \frac{\pi}{2}$ instead of ϕ;

$$\therefore r'^2 = a^2\sin^2\phi + b^2\cos^2\phi,$$

$$\therefore r^2 + r'^2 = a^2 + b^2.$$

Again,

$$\sin(\theta' - \theta) = \sin\theta'\cos\theta - \cos\theta'\sin\theta$$

$$= \frac{y'}{r'}\frac{x}{r} - \frac{x'}{r'}\frac{y}{r};$$

$$\begin{aligned} \therefore rr'\sin(\theta' - \theta) &= y'x - x'y \\ &= ab(\sin\phi'\cos\phi - \cos\phi'\sin\phi) \\ &= ab. \end{aligned}$$

since $\quad \phi' - \phi = \dfrac{\pi}{2}$,

which are two well known properties of conjugate diameters.

(3) If in the expressions $x = a\cos\phi$, $y = b\sin\phi$, we put $\phi + \frac{\pi}{2}$ for ϕ', we change x and y into x' and y', therefore

$$x' = -a\sin\phi = -\frac{a}{b}y,$$

$$y' = \quad b\cos\phi = \quad \frac{b}{a}x.$$

(4) To find the evolute to the ellipse, the equation to the normal being

$$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1),$$

may be put in the form

$$y - b \sin \phi = \frac{a \sin \phi}{b \cos \phi} (x - a \cos \phi),$$

$$\text{or} \quad y = x \cdot \frac{a}{b} \tan \phi - \frac{a^2 - b^2}{b} \sin \phi.$$

Therefore, differentiating with respect to ϕ, we have

$$0 = x \frac{a}{b} \frac{1}{\cos^2 \phi} - \frac{a^2 - b^2}{b} \cos \phi;$$

$$\therefore \cos^3 \phi = \frac{x}{\alpha} \quad \left(\alpha = \frac{a^2 - b^2}{b}\right),$$

and changing ϕ, a, x, into $\frac{\pi}{2} - \phi$, b, y,

$$\sin^3 \phi = \frac{y}{\beta} \quad \left(\beta = \frac{b^2 - a^2}{a}\right),$$

$$\therefore \left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1.$$

(5) By substituting $x = a \cos \phi$, $y = b \sin \phi$, in the equation of areas in central forces, viz.

$$(x - ae) \frac{dy}{dt} - y \frac{dx}{dt} = h,$$

we find immediately

$$ab \left\{(\cos \phi - e) \cos \phi + \sin^2 \phi\right\} d\phi = hdt,$$

$$\text{or} \quad (1 - e \cos \phi) \, d\phi = ndt \left(n = \frac{h}{ab}\right),$$

$$\therefore \phi - e \sin \phi = nt + \text{const.}$$

These few simple examples shew the use of the eccentric anomaly.

(6) To determine the magnitude and position of the axes of the conic section

$$Ax^2 + 2Bxy + Cy^2 = 1 \quad \ldots\ldots\ldots\ldots \text{(1)},$$

the co-ordinates being oblique.

Let ω be the angle of ordination, then the equation to a circle having the origin as centre is

$$x^2 + 2xy \cos \omega + y^2 = r^2 \quad \ldots\ldots\ldots\ldots \text{(2)}.$$

$$\text{(1)}\, r^2 - \text{(2) gives}$$

$$(Ar^2 - 1) + 2 (Br^2 - \cos \omega) \frac{y}{x} + (Cr^2 - 1) \frac{y^2}{x^2} = 0 \ldots\ldots \text{(3)}.$$

If this equation gives two equal values of $\frac{y}{x}$, the circle (2) must touch the conic section (1), which can only be at the extremities of the axes. Therefore the condition that (3) may have equal roots, namely,

$$(Ar^2 - 1)(Cr^2 - 1) = (Br^2 - \cos\omega)^2 \; \ldots\ldots \; (4),$$

makes r^2 equal to either a^2 or b^2.

Moreover, if z be the value of $\frac{y}{x}$ got from (3), the equation to either of the axes is

$$y = zx \; \ldots\ldots\ldots\ldots\ldots\ldots \; (5).$$

Hence a^2 and b^2 are the roots of

$$(AC - B^2)\, r^4 - (A + C + 2B\cos\omega)\, r^2 + \sin^2\omega = 0 \ldots (6);$$

and to determine the positions of the axes, since (3) is a perfect square, we have

$$(Cr^2 - 1)\frac{y}{x} + Br^2 - \cos\omega = 0,$$

$$(Ar^2 - 1)\frac{x}{y} + Br^2 - \cos\omega = 0.$$

Eliminating r^2 from these two equations, we find

$$(Cy - Bx)(x - y\cos\omega) - (Ax - By)(y - x\cos\omega) = 0,$$

which, by (5), is evidently the equation to the two axes considered as one locus.

(7) If $B = 0$ in (6), we have

$$r^4 - \left(\frac{1}{A} + \frac{1}{C}\right) r^2 + \frac{\sin^2\omega}{AC} = 0;$$

therefore, if we put $\frac{1}{A} = a'^2$, $\frac{1}{C} = b'^2$, we find

$$a^2 + b^2 = a'^2 + b'^2,$$

$$ab = a'b' \sin\omega.$$

M. O. B.

III.—ON THE THEORY OF ALGEBRAIC CURVES.

By A. Cayley, B.A. Fellow of Trinity College.

Suppose a curve defined by the equation $U = 0$, U being a rational and integral function of the m^{th} order of the co-ordinates x, y. It may always be assumed, without loss of generality, that the terms involving x^m, y^m, both of them appear in (U); and also that the coefficient of y^m is equal to unity. For in any particular curve where this was not the case, by transforming the axes, and dividing the new equation by the

coefficient of y^m, the conditions in question would become satisfied. Let H_m denote the terms of U, which are of the order (m), and let $y-\alpha x$, $y-\beta x \ldots y-\lambda x$ be the factors of H_m. If the quantities $\alpha, \beta \ldots \lambda$ are all of them different, the curve is said to have a number of asymptotic directions equal to the degree of its equation. Such curves only will be considered in the present paper, the consideration of the far more complicated theory of those curves, the number of whose asymptotic directions is less than the degree of their equation, being entirely rejected. Assuming, then, that the factors of H_m are all of them different, we may deduce from the equation $U=0$, by known methods, the series

$$y = \alpha x + \alpha' + \frac{\alpha''}{x} + \ldots \qquad (1).$$

$$y = \beta x + \beta' + \frac{\beta''}{x} + \ldots$$

$$\vdots$$

$$y = \lambda x + \lambda' + \frac{\lambda''}{x} + \ldots$$

And these being obtained, we have, identically,

$$U = (y - \alpha x - \alpha' - \ldots)(y - \beta x - \beta' - \ldots)\ldots(y - \lambda x - \lambda' - \ldots)\ldots(2).$$

The negative powers of x on the second side, in point of fact, destroying each other. Supposing in general that fx containing positive and negative powers of x, Efx denotes the function which is obtained by the rejection of the negative powers, we may write

$$U = E\left(y - \alpha x . . - \frac{\alpha^{(m)}}{x^{m-1}}\right)\left(y - \beta x . . - \frac{\beta^{(m)}}{x^{m-1}}\right)\ldots\left(y - \lambda x . . - \frac{\lambda^{(m)}}{x^{m-1}}\right)\ldots(3),$$

the symbol E being necessary in the present case, because, when the series are continued only to the power x^{-m+1}, the negative powers no longer destroy each other.

We may henceforward consider U as originally given by the equation (3), the $m.(m+1)$ quantities $\alpha, \alpha' . . \alpha^{(m)}$, $\beta, \beta' . . \beta^{(m)}$, $. . \lambda, \lambda' . . \lambda^{(m)}$ satisfying the equations obtained from the supposition that it is possible to determine the following terms $\alpha^{(m+1)}$, $\beta^{(m+1)} . . \lambda^{(m+1)}, . .$ so that the terms containing negative powers of x, on the second side of equation (2), vanish. It is easily seen that $\alpha, \beta . . \lambda$, $\alpha', \beta' . . \lambda'$ are entirely arbitrary, $\alpha'', \beta'' . . \lambda''$ satisfy a single equation involving only the preceding quantities, $\alpha''', \beta''' . . \lambda'''$ two equations involving the quantities which precede them, and so on, until $\alpha^{(m)}, \beta^{(m)} \ldots \lambda^{(m)}$, which satisfy $(m-1)$ relations, involving the preceding quan-

tities. Thus the $m.(m+1)$ quantities in question satisfy $\frac{m.m-1}{2}$ equations, or they may be considered as functions of $m.(m+1) - \frac{1}{2}m.(m-1) = \frac{1}{2}m.(m+3)$ arbitrary constants. Hence the value of U, given by the equation (3), is the most general expression for a function of the m^{th} order. It is to be remarked also that the quantities $\alpha^{(m+1)}, \beta^{(m+1)}..\lambda^{(m+1)},....$ are all of them completely determinable as functions of $\alpha, \beta..\lambda, ..\alpha^{(m)}, \beta^{(m)}..\lambda^{(m)}$.

The advantage of the above mode of expressing the function U, is the facility obtained by means of it for the elimination of the variable (y) from the equation $U=0$, and any other analogous one $V=0$. In fact, suppose V expressed in the same manner as U, or by the equation

$$V=E\left(y-Ax...-\frac{A^{(n)}}{x^{n-1}}\right)\left(y-Bx...-\frac{B^{(n)}}{x^{n-1}}\right)...\left(y-Kx...-\frac{K^{(n)}}{x^{n-1}}\right)...(4),$$

n being the degree of the function V. It is almost unnecessary to remark, that $A, B..K,..A^{(n)}, B^{(n)}..K^{(n)}$ are to be considered as functions of $\frac{1}{2}n(n+3)$ arbitrary constants, and that the subsequent $A^{(n+1)}, B^{(n+1)}..K^{(n+1)}..$can be completely determined as functions of these. Determining the values of y from the equation (3), viz. the values given by the equations (2); substituting these successively in the equation

$$V=(y-Ax-..)(y-Bx-..)..(y-Kx-..)=0...(5),$$

analogous to (2), and taking the product of the quantities so obtained, also observing that this product must be independent of negative powers of x, the result of the elimination may be written down under the form

$$E\left[\begin{array}{l}\left\{(\alpha-A)x..+\dfrac{\alpha^{(mn)}-A^{(mn)}}{x^{mn-1}}\right\}...\left\{(\alpha-Kx)...+\dfrac{\alpha^{(mn)}-K^{(mn)}}{x^{mn-1}}\right\}\\ \times\left\{(\lambda-A)x..+\dfrac{\lambda^{(mn)}-A^{(mn)}}{x^{mn-1}}\right\}...\left\{(\lambda-Kx)...+\dfrac{\lambda^{(mn)}-K^{(mn)}}{x^{mn-1}}\right\}\end{array}\right]...(6),$$

the series in $\{\ \}$ being continued only to x^{-mn+1}, because the terms after this point produce in the whole product nothing but terms involving negative powers of x. It is for the same reason that the series in ( ), in the equations (3) and (4), are only continued to the terms involving x^{-m+1}, x^{-n+1} respectively.

The first side of the equation (6) is of the order mn, in x, as it ought to be. But it is easy to see, from the form of the expression, in what case the order of the first side reduces

itself to a number less than mn. Thus, if n be not greater than m, and the following equations be satisfied,

$$\begin{aligned}
&A = a,\ A^{(1)} = a^{(1)} \ldots A^{(r-1)} = a^{(r-1)}\ r \not> n \ldots\ldots (7).\\
&B = \beta,\ B^{(1)} = \beta^{(1)} \ldots B^{(s-1)} = \beta^{(s-1)}\ s \not> r\\
&\vdots\\
&K = \kappa,\ K^{(1)} = \kappa^{(1)} \ldots K^{(v-1)} = \kappa^{(v-1)}\ v \not> u.
\end{aligned}$$

The degree of the equation (6) is evidently $mn - r - s \ldots - v$, or the curves $U = 0$, $V = 0$ intersect in this number only of points. If $mn - r - s \ldots - v = 0$. The curves $U = 0$ and $V = 0$ do not intersect at all, and if $mn - r - s - v$ be negative, $= -\omega$ suppose, the equation (6) is satisfied identically, or the functions U, V have a common factor, the number ω expressing the degree of this factor in x, y.

Supposing the function V given arbitrary, it may be required to determine U, so that the curves $U = 0$, $V = 0$ intersect in a number $mn - k$ points. This may in general be done, and done in a variety of ways, for any value of k from unity to $\frac{1}{2} m.(m+3)$. I shall not discuss the question generally at present, nor examine into the meaning of the quantity $mn - \frac{1}{2} m \,.\, m + 3 \ \{= \frac{1}{2} m \,.\, (2n - m - 3)\}$ becoming negative, but confine myself to the simple case of U and V, both of them functions of the second order. It is required, then, to find the equations of all those curves of the second order which intersect a given curve of the second order in a number of points less than four.

Assume, in general,

$$V = E\left(y - Ax - A' - \frac{A''}{x}\right)\left(y - Bx - B' - \frac{B''}{x}\right),$$

A'', B'' satisfy $A'' + B'' = 0$, and putting $B'' = \dfrac{K}{A - B}$, and $\therefore A'' = -\dfrac{K}{A - B}$, and reducing

$$V = (y - Ax - A')(y - Bx - B') + K.$$

Similarly $U = E\left(y - ax - a' - \dfrac{a''}{x}\right)\left(y - \beta x - \beta' - \dfrac{\beta''}{x}\right)$,

$a'' + \beta'' = 0$, and putting $\beta'' = \dfrac{k}{a - \beta}$, and $\therefore a'' = -\dfrac{k}{a - \beta}$, and reducing

$$U = (y - ax - a')(y - \beta x - \beta') + k.$$

Suppose

(1) $U = 0$, $V = 0$ intersect in three points, we must have $a = A$, or the curve $U = 0$ must have one of its asymptotes parallel to one of the asymptotes of $V = 0$.

(2) The curves intersect in two points. We must have $\alpha = A$, $\alpha' = A'$, or else $\alpha = A$, $\beta = B$; i.e. $U = 0$ must have one of its asymptotes coincident with one of the asymptotes of the curve $V = 0$, or else it must have its asymptotes each of them parallel to $V = 0$. The latter case is that of similar and similarly situated curves.

(3) Suppose the curves intersect in a single point only. Then either $\alpha = A$, $\alpha' = A'$, $\alpha'' = A''$, which it is easy to see gives

$$U = (y - Ax - A')(y - \beta x - \beta') + K.\frac{A - \beta}{A - B},$$

or else $\alpha = A$, $\alpha' = A'$, $\beta = B$, which is the case of one of the asymptotes of the curve $U = 0$, coinciding with one of those of the curve $V = 0$, and the remaining asymptotes parallel. As for the first case, if a, a_1 are the transverse axes, θ, θ_1 the inclination of the two asymptotes to each other, these four quantities are connected by the equation

$$\frac{a^2}{a_1^2} = \frac{\tan\theta\,.\cos^2\dfrac{\theta}{2}}{\tan\theta_1\cos^2\dfrac{\theta_1}{2}};$$

and besides, one of the asymptotes of the first curve is coincident with one of the asymptotes of the second. This is a remarkable case; it may be as well to verify that $U = 0$, $V = 0$ do intersect in a single point only. Multiplying the first by $y - Bx - B'$, the second by $y - \beta x - \beta'$, and subtracting, the result is

$$(A - \beta)(y - Bx - B') - (A - B)(y - \beta x - \beta') = 0,$$

reducible to

$$y - Ax = \frac{A(B' - \beta') + B\beta' - B'\beta}{B - \beta}, \text{ i.e. } y - Ax - C = 0.$$

Combining this with $V = 0$, we have an equation of the form $y - Bx - D = 0$. And from this and $y - Ax - C$, x, y are determined by means of a simple equation.

(4) Lastly, when the curves do not intersect at all. Here $\alpha = A$, $\alpha' = A'$, $\beta = B$, $\beta' = B'$, or the asymptotes of $U = 0$ coincide with those of $V = 0$; *i.e.* the curves are similar, similarly situated, and concentric; or else $\alpha = A$, $\alpha' = A'$, $\alpha'' = A''$, $\beta = B$. Here

$$U = (y - Ax - A')(y - Bx - \beta') + K,$$

or the required curve has one of its asymptotes coincident with one of those of the proposed curve; the remaining two

asymptotes are parallel, and the magnitudes of the curves are equal.

In general, if two curves of the orders m and n, respectively, are such that r asymptotes of the first are parallel to as many of the second, s out of these asymptotes being coincident in the two curves, the number of points of intersection is $mn - r - s$; but the converse of this theorem is not true.

In a former paper, *On the Intersection of Curves*, I investigated the number of arbitrary constants in the equation of a curve of a given order (ρ) subjected to pass through the mn points of intersection of two curves of the orders m and n respectively. The reasoning there employed is not applicable to the case where the two curves intersect in a number of points less than mn. In fact, it was assumed that, $W = 0$ being the equation of the required curve, W was of the form $uU + vV$; u, v being polynomials of the degrees $\rho - m$, $\rho - n$ respectively. This is, in point of fact, true in the case there considered, viz. that in which the two curves intersect in mn points; but where the number of points of intersection is less than this, u, v may be assumed polynomials of an order *higher* than $\rho - m$, $\rho - n$, and yet $uU + vV$ reduce itself to the order (ρ). The preceding investigations enable us to resolve the question for every possible case.

Considering then the functions U, V determined as before by the equations (3), (4). Suppose, in the first place, we have a system of equations

$$\alpha = A,\ \beta = B. \ldots \theta = H \ (t \text{ equations}) \ldots. \quad (8).$$

Assume $P = (y - \alpha x - \ldots)(y - \beta x - \ldots)\ldots(y - \theta x - \ldots)$,

$$Q = (y - Ax - \ldots)(y - Bx - \ldots)\ldots(y - Hx - \ldots).$$

$$\Upsilon = (y - \iota x - \ldots)\ldots(y - \lambda x - \ldots),$$

$$\Psi = (y - \mathrm{I}x\ldots)\ldots(y - Kx - \ldots);$$

whence $\qquad U = P\Upsilon, \quad V = Q\Psi.$

Suppose $\qquad \Upsilon = E\Upsilon + \Delta\Upsilon, \quad \Psi = E\Psi + \Delta\Psi,$

$$\begin{aligned} E\Psi.U - E\Upsilon.V &= E\Psi.P\Upsilon - E\Upsilon.Q\Psi, \\ &= E\Psi.P.(E\Upsilon + \Delta\Upsilon) - E\Upsilon.Q\,(E\Psi + \Delta\Psi), \\ &= E\Upsilon.E\Psi.(P - Q) + E\Psi.P.\Delta\Upsilon - E\Upsilon.Q.\Delta\Psi, \\ &= E\,\{E\Upsilon.E\Psi.(P - Q) + E\Psi.P\Delta\Upsilon - E\Upsilon.Q.\Delta\Psi\}, \\ &= \Pi \ \text{suppose}. \end{aligned}$$

In this expression $E\Upsilon$, $E\Psi$ are of the degrees $m - t$, $n - t$, $\Delta\Upsilon$, $\Delta\Psi$ of the degree (-1), and P, Q, $P - Q$ of the degrees t, t, $(t - 1)$ respectively. The terms of Π are therefore of the

degrees $m+n-t-1$, $(m-1)$, $(n-1)$ respectively, and the largest of these is in general $m+n-t-1$. Suppose, however, $m+n-t-1$ should be equal to $(m-1)$ (it cannot be inferior to it), then $t=n$. Ψ becomes equal to unity, or $\Delta\Psi$ vanishes. The remaining two terms of Π are $E\Upsilon\,(P-Q)$, $P\Delta\Upsilon$, which are of the degrees $(m-1)$, $(n-1)$ respectively, Π is still of the degree $(m-1)$, supposing $m>n$. If $m=n$, the term $P\Delta\Upsilon$ vanishes. Π is still of the degree $m-1$. Hence in every case the degree of Π is $m+n-t-1$. (Assuming always that $(P-Q)$ does not reduce itself to a degree lower than $t=1$, which is always the case as long as the equations $\alpha'=A'$, $\beta'=B'\ldots.\theta'=H'$ are not all of them satisfied simultaneously). It will be seen presently that we shall gain in symmetry by wording the theorem thus: the degree of Π is equal to the greatest of the two quantities $m+n-t-1$, $m-1$.

Suppose next, in addition to the equations (8), we have

$$\alpha'=A',\ \beta'=B'\ldots.\zeta'=F',\quad t' \text{ equations } t' \not> t\ldots(8').$$

Then, taking Υ', Ψ', P', Q' the analogous quantities to Υ, Ψ, P, Q, and putting $E\Psi'.U-E\Upsilon'.V.=\Pi'$,

we have, as before,

$$\Pi'=E\{E\Upsilon'.E\Psi'.(P'-Q')+E\Psi'.P'\Delta\Upsilon'-E\Upsilon'.Q'\Delta\Psi'.\}.$$

The degree of $P'-Q'$ is $t'-2$ (unless simultaneously $\alpha''=A''$, $\beta''=B''\ldots.\zeta''=F''$, in which case the degree may be lower). The degrees, therefore, of the terms of Π' are $m+n-t'-2$, $n-1$, $m-1$. Or we may say that the degree of Π' is equal to the greatest of the quantities $m+n-t'-2$, $m-1$; though to establish this proposition in the case where $t'=(n-1)$ would require some additional considerations.

Continuing in this manner until we come to the quantity $\Pi^{(k-1)}$, the degree of this quantity is the greatest of the two numbers $m+n-t^{(k-1)}-k$, $(m-1)$. And we may suppose that none of the equations $\alpha^{(k)}=A^{(k)}\ldots.$ are satisfied, so that the series Π, $\Pi'\ldots.\Pi^{(k-1)}$ is not to be continued beyond this point.

Considering how the equation of the curve passing through the $mn-t-t'\ldots-t^{(k-1)}$ points of intersection of $U=0$, $V=0$. We may write

$$W=uU+vV+p\Pi+p'\Pi'\ldots.+p^{(k-1)}\,\Pi^{(k-1)}=0\ldots.(9)$$

for the required equation; the dimensions of $u, v, p, p'\ldots p^{(k-1)}$ being respectively

$$\rho-m,\ \rho-n,\ \rho-m-n+t+1 \text{ or } \rho-m+1,\ \rho-m-n+t'+2$$
$$\text{or } \rho-m+1,\ldots.\ \rho-m-n+t^{(k-1)}+k \text{ or } \rho-m+1,$$

the lowest of the two numbers being taken for the dimensions of $p, p' \ldots p^{k-1}$. Also, if any of these numbers become negative, the corresponding term is to be rejected. In saying that the degrees of $p, p' \ldots p^{(k-1)}$ have these actual values, it is supposed that the degrees of $\Pi, \Pi' \ldots \Pi^{k-1}$ actually ascend to the greatest of the values

$\rho - m - n + t + 1$, or $(m-1)$, $m + n - t' - 2$, or $(m-1)$, $-m + n - t^{(k-1)} + k$, or $(m-1)$.

The cases of exception to this are when several of the consecutive numbers $t, t' \ldots t^{(k-1)}$ are equal. In this case the corresponding terms of the series $\Pi, \Pi' \ldots \Pi^{k-1}$, are also equal. Suppose for instance t, t' were equal, Π, Π' would also be equal. A term of p of an *order higher* by unity than $\rho - m - n + t + 1$, or $\rho - m + 1$, which is the highest term admissible, produces in $p\Pi$ a term, identical with one of the terms of $p'\Pi$. So that nothing is gained in generality by admitting such terms into p. The equation (9), with the preceding values for the dimensions of $p, p' \ldots p^{(k-1)}$, may be employed, therefore, even when several consecutive terms of the sines $t, t' \ldots t^{(k-1)}$ are equal. It will be convenient also to assume that $\rho - m$ is not negative, or at least for greater simplicity to examine this case in the first place.

u, U, and v, V, contain terms of the form $x^\alpha y^\beta U$, $x^\gamma y^\delta V$, $\alpha + \beta \not> \rho - m$, $\gamma + \delta \not> \rho - n$. $p\Pi$ contains terms of this form, and m addition terms where $\alpha + \beta = (\rho + 1 - m)\, \gamma + \delta = (\rho + 1 - n)$. It is useless to repeat the former terms, so we may assume for p, a *homogeneous* function of the order $\rho - m - n + t + 1$, or $\rho - m + 1$; in which case $p\Pi$ consists only of terms where $\alpha + \beta = (\rho + 1 - m)$, $\gamma + \delta = (\rho + 1 - n)$. And the general expression of p contains $\rho - m - n + t + 2$, or $\rho - m + 2$, arbitrary constants. Similarly $p'\Pi'$ contains terms of the form of those in uU, vV, $p\Pi$, and also terms for which

$$\alpha + \beta = (\rho + 2 - m),\ \gamma + \delta = \rho + 2 - n.$$

The latter terms only need be considered, or p' may be assumed to be a homogeneous function of the order $\rho - m - n + t' + 2$, or $\rho - m + 1$, containing therefore $\rho - m - n - t' + 3$, or $(\rho - m) + 2$ arbitrary constants.

Similarly $p^{(k-1)}$ contains $\rho - m - n + t^{(k-1)} + k + 1$ or $\rho - m + 2$ arbitrary constants. Assume

$$\nabla = \begin{pmatrix} \rho-m-n+t+2 \\ \rho-m+2 \end{pmatrix} + \begin{pmatrix} \rho-m-n+t'+3 \\ \rho-m+2 \end{pmatrix} \ldots + \begin{pmatrix} \rho-m-n+t^k+k+1 \\ \rho-m+2 \end{pmatrix} \ldots (10),$$

where, in forming the value of ∇ the least of the two quantities in ( ) is to be taken, this value also, if negative, being replaced by zero. The number of arbitrary constants in $p, p' \ldots p^{(k-1)}$ is consequently equal to Δ.

The number of arbitrary constants in u, v, are respectively $\{1+2.\ldots.(\rho-m+1)\}$ and $\{1+2.\ldots.(\rho-n+1)\}$ i.e.

$$\tfrac{1}{2}(\rho-m+1)(\rho-m+2), \text{ and } \tfrac{1}{2}(\rho-n+1)(\rho-n+2).$$

Or the whole number of arbitrary constants in W, diminished by unity (since nothing is gained in generality, by leaving the coefficient, for instance of y^ρ indeterminate, instead of supposing it equal to unity) becomes

$$\tfrac{1}{2}(\rho-m+1)(\rho-m+2)+\tfrac{1}{2}.(\rho-n+1)(\rho-n+2)+\nabla-1,$$

reducible to

$$\tfrac{1}{2}\rho.(\rho+3)+\tfrac{1}{2}.(\rho-m-n+1)(\rho-m-n+2)-mn+\nabla.$$

By the reasonings contained in the paper already referred to, if $\rho+k-m-n+1$ be positive, to find the number of really disposable constants in W, we must subtract from this number a number $\frac{1}{2}(\rho+k-m-n+1)(\rho+k-m-n+2)$. Hence, calling ϕ the number of disposable constants in W, we have

$$\phi=\tfrac{1}{2}\rho.(\rho+3)+\tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2)-mn+\nabla-\Lambda \ldots (11),$$

where $\Lambda=0$, if $\rho+k-m-n+1$ be negative or zero ... (12),

$$\Lambda=\tfrac{1}{2}(\rho+k-m-n+1)(\rho+k-m-n+2),$$

if $\rho+k-m-n+1$ be positive.

And ∇ is given by the equation (9). Also, if θ be the number of points through which the curve $W=0$ can be drawn, *including* the points of intersection of the curves $U=0$, $V=0$, $\theta=\phi+(mn-t-t'.\ldots.-t^{k-1})$ or

$$\theta=\tfrac{1}{2}\rho.(\rho+3)+\tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2)+\nabla-\Lambda-t-t'\ldots-t^{k-1}\ldots (13).$$

Any particular cases may be deduced with the greatest facility from these general formulæ. Thus, supposing the curves to intersect in the complete number of points mn, we have

$$\phi=\tfrac{1}{2}\rho.(\rho+3)+\tfrac{1}{2}(1-\aleph)(\rho-m-n+1)(\rho-m-n+2)-mn,$$

$\aleph$ being zero or unity according as $\rho<(m+n-1)$ or $\rho>(m+n-1)$. Reducing, we have, for $\rho \ngtr m+n-3$,

$$\phi=\tfrac{1}{2}\rho.(\rho+3)+\tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2)-mn,$$
$$\theta=\tfrac{1}{2}\rho.(\rho+3)+\tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2).$$

And for $\rho>m+n-3$,

$$\phi=\tfrac{1}{2}\rho.(\rho+3)-mn,$$
$$\theta=\tfrac{1}{2}\rho.(\rho+3).$$

Suppose, in the next place, the curves have t parallel pairs of asymptotes, none of these pairs being coincident. Then $\rho \ngtr m+n-t-2$,

$$\phi=\tfrac{1}{2}\rho.(\rho+3)+\tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2)-mn,$$
$$\theta=\tfrac{1}{2}\rho.(\rho+3)+\tfrac{1}{2}(\rho-m-n+1)(\rho-m-n+2)-t.$$

$\rho > m + n - t - 2, \rho \not> m + n - 2,$

$$\phi = \tfrac{1}{2}\rho.(\rho + 3) + \tfrac{1}{2}(\rho - m - n + 2)(\rho - m - n + 3) - mn + t,$$

$$\theta = \tfrac{1}{2}\rho.(\rho + 3) + \tfrac{1}{2}(\rho - m - n + 2)(\rho - m - n + 3),$$

$\rho > m + n - 2, \quad \phi = \tfrac{1}{2}.\rho.(\rho + 3) - mn + t,$

$$\theta = \tfrac{1}{2}\rho.(\rho + 3).$$

In which, if t be equal to 2 or greater than 2, the limiting conditions are more conveniently written

$\rho \not> m+n-t-2\,;\ \rho \not> m+n-t-2 > m+n-4\,;\ \rho > m+n-4.$

Similarly may the solution of the question be explicitly obtained when the curves have t asymptotes parallel, and t' out of these coincident, but the number of separate formulæ will be greater.

In conclusion, I may add the following references to two memoirs on the present subject: the conclusions in one point of view are considerably less general even than those of my former paper, though much more so in another. Jacobi Theoremata de punctis intersect. duar. curvar. algeb.; Crelle's Journal, vol. XV.; Plucker Theoremes sur les equations a plusieurs variables, d° vol. XVI.

Addition.

As an exemplification of the preceding formulæ, and besides, as a question interesting in itself, it may be proposed to determine the asymptotic curves of the r^{th} order of a given curve, having all its asymptotic directions distinct,—r being any number less than the degree of the equation of the given curve.

Def$^{\text{n}}$. A curve of the r^{th} order, which intersects a given curve of the m^{th} order in a number of points, $= mr - \dfrac{r.(r+3)}{2}$, is said to be an asymptotic curve of the r^{th} order to the curve in question. Suppose, as before, $U = 0$ being the equation to the given curve,

$$U = E\left(y - \alpha x - \ldots - \frac{\alpha^{(m)}}{x^{m-1}}\right)\ldots\left(y - \lambda x\ldots - \frac{\lambda^{(n)}}{x^{m-1}}\right);$$

and let $\theta, \phi..\omega$ denote any combination of r terms out of the series $\alpha..\lambda..\theta', \phi'..\omega'$, &c...the corresponding terms out of $\alpha'..\lambda'$, &c. Then, writing

$$V = E\left\{\left(y - \theta x\ldots - \frac{\theta^{(m)}}{x^{m-1}}\right)\left(y - \phi x\ldots - \frac{\phi^{(m-1)}}{x^{m-2}} - \frac{\Phi^{(m)}}{x^{m-1}}\right)\times\ldots.\right.$$

$$\left.\left(y - \psi x\ldots - \frac{\psi^{(m-2)}}{x^{m-3}} - \frac{\Psi^{(m-1)}}{x^{m-2}} - \frac{\Psi^{(m)}}{x^{m-1}}\right)\ldots\left(y - \omega x - \Omega'\ldots - \frac{\Omega^{(m)}}{x^{m-1}}\right)\right\},$$

(where the quantities $\Phi^{(m)}$, $\Psi^{(m-1)}$, $\Psi^{(m)}$. .Ω'. .$\Omega^{(m)}$ are entirely determinate, since, by what has preceded, θ', ϕ'. .Ω' satisfy a certain equation θ'', ϕ''. .Ω'' two equations. . . .$\theta^{(m)}$ $\Phi^{(m)}$. .$\Omega^{(m)}$ $(m-1)$ equations), we have

$$V=0$$

for the required equation of the asymptotic curve. It is obvious that the whole number of asymptotic curves of the order r, is $n.(n-1)..(n-r+1)$, viz. $1.2..r$ curves for each combination of $\frac{n.(n-1)..(n-r+1)}{1.2..r}$ asymptotes. Some particular instances of asymptotic curves will be found in a memoir by M. Plucker, *Liouville's Journal*, vol. I. on the Enumeration of Curves of the Fourth Order. The general theory does not seem to be one to which much attention has been paid.

IV.—THE POLAR EQUATION TO THE TANGENT TO A CONIC SECTION.

LET the polar equation to the conic section be

$$\frac{c}{r}=1+e\cos\theta.$$

Let the equation to the tangent be

$$\frac{c}{r}=m\cos\theta+n\sin\theta\,;$$

therefore at the point of contact the two values of $\sin\theta$, given by the equation

$$1+e\cos\theta=m\cos\theta+n\sin\theta,$$

are each $=\sin\alpha$, if α be the value of θ at the point of contact; and because $(1-n\sin\theta)^2=(m-e)^2(1-\sin^2\theta)$, by the condition of equal roots,

$$\{n^2+(m-e)^2\}\,\{1-(m-e)^2\}=n^2,$$
$$\therefore\ (m-e)^2+n^2=1,$$
$$\therefore\ \sin\alpha-n=0,$$
$$\cos\alpha-(m-e)=0\,;$$

and the equation to the tangent is

$$\frac{c}{r}=e\cos\theta+\cos(\theta-\alpha).$$

If (fig. 1) PT, QT be tangents at points P and Q of a conic section whose equation is $\frac{c}{r}=1+e\cos\theta$; α, β the

spiral angles at P and Q, reckoned from the pole, the equations to PT, QT are

$$\frac{c}{r} = \cos(\theta - \alpha) + e \cos\theta,$$

$$\text{and } \frac{c}{r} = \cos(\theta - \beta) + e \cos\theta;$$

therefore at the point of intersection

$$\cos(\theta - \alpha) = \cos(\theta - \beta),$$

$$\theta - \alpha = \beta - \theta,$$

or ST bisects the angle PSQ.

To find the locus of the intersection of a perpendicular to the focal distance from the focus with the tangent.

In this case the equation to the tangent is

$$\frac{c}{r} = \cos(\theta - \alpha) + e \cos\theta,$$

equation to the perpendicular supposed

$$\theta = \alpha - \frac{\pi}{2};$$

therefore, eliminating α, the equation to the locus required is

$$\frac{c}{r} = e \cos\theta,$$

$$\text{or } r\cos\theta = \frac{c}{e},$$

the equation to the directrix.

To find the locus of the intersection of two tangents at extremities of focal distances which make equal angles with the latus rectum.

Here $\alpha = \pi - \beta,$

$$\theta = \frac{\alpha + \beta}{2} = \frac{\pi}{2},$$

or the locus is the latus rectum.

(Fig. 2) The enunciation of this problem is given in p. 171, vol. I. The axes are $AKL . ARQ$.

Let the equation to HK be $a_1x + b_1y = 1$ ....(1),

............HL....$a_2x + b_2y = 1$..........(2),

............PQ....$\alpha_1x + \beta_1y = 1$..........(3),

............PR....$\alpha_2x + \beta_2y = 1$..........(4),

............KQ....$a_1x + \beta_1y = 1$..........(5),

............RL....$a_2x + \beta_2y = 1$..........(6),

where a_1 b_1 &c. are the reciprocals of the intercepts for the different lines. H, O, and P are in a straight line.

The co-ordinates of H are given by (1) and (2),

.... P............ (3) and (4),

.... O............ (5) and (6).

Multiply (1) and (5) by h,
(2) and (6) by k.

Then, by subtraction, we have

$$\text{at } H \ (a_1h - a_2k)\,x + (b_1h - b_2k)\,y = h - k,$$

$$\text{at } O \ (a_1h - a_2k)\,x + (\beta_1h - \beta_2k)\,y = h - k;$$

$$\text{and if } b_1h - b_2k = \beta_1h - \beta_2k,$$

$$\text{or } (b_1 - \beta_1)\,h = (b_2 - \beta_2)\,k,$$

the relations are the same, or either is the equation to the straight line joining HO;

$$\text{at } P \ (a_1h - \beta_1k)\,x + (\beta_1h - \beta_2k)\,y = h - k,$$

$$\therefore \ a_1h - a_2k = \alpha_1h - \alpha_2k,$$

$$\text{or} \quad (a_1 - \alpha_1)\,h = (a_2 - \alpha_2)\,k;$$

$$\therefore \ \frac{a_1 - \alpha_1}{b_1 - \beta_1} = \frac{a_2 - \alpha_2}{b_2 - \beta_2} \ \ldots\ldots\ldots\ldots (A).$$

Again, at C, $\quad (a_1 - \alpha_1)\,x + (b_1 - \beta_1)\,y = 0,$

at B, $\quad (a_2 - \alpha_2)\,x + (b_2 - \beta_2)\,y = 0,$

which two equations, by (A), are equations to the same straight line which passes through the origin A.

If $ABCD$ (fig. 3) be any conic section, AB any chord of the conic section, $ACBD$ a quadrilateral on AB, P the intersection of AD, BC, Q the intersection of the diagonals; PQ always passes through the point O, which is the intersection of two tangents.

Let $\quad OA = \dfrac{1}{a}, \quad OB = \dfrac{1}{b};$

equation to AD $\quad ax + \beta y = 1$............(1),

.... BC $\quad \gamma x + by = 1$............(2),

.... BD $\quad \alpha x + by = 1$............(3),

.... AC $\quad ax + \delta y = 1$............(4);

equation to the conic section,

$$(ax + by - 1)^2 = Babxy,$$

$$\text{or} \quad ax + by - 1 = \sqrt{(Babxy)};$$

at D the intersection of (1) and (3)

$$\alpha x + \beta y - 1 = 0;$$

therefore $(b - \beta)\, y = \sqrt{(Babxy)}$,

or $(b - \beta)^2\, y = Babx$.

Also $(a - \alpha)\, x = (b - \beta)\, y$,

therefore $(a - \alpha)(b - \beta) = Bab$.

Similarly $(a - \gamma)(b - \delta) = Bab$,

$$\text{Hence} \quad \frac{a - \alpha}{b - \delta} = \frac{a - p}{b - \beta} \quad \ldots\ldots\ldots\ldots (A).$$

Also at P $(a - \gamma)\, x = (b - \beta)\, y$,

at Q $(a - \alpha)\, x = (b - \delta)\, y$;

which two relations coincide by (A), and either is the equation to PQ, which therefore passes through O.

V.—DEMONSTRATION OF A PROPOSITION IN PHYSICAL OPTICS.

To shew that the planes of polarization of the waves in a biaxal crystal bisect the angles between the planes through the normal to the front and the wave axes.

If l, m, n be the direction cosines of a wave axis, the axes of elasticity being the axes of co-ordinates, it is shewn that

$$l = \pm \sqrt{\left(\frac{a^2 - b^2}{a^2 - c^2}\right)}, \quad m = 0, \quad n = \sqrt{\left(\frac{b^2 - c^2}{a^2 - c^2}\right)}.$$

Let $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ be the direction cosines of the lines of rectilinear vibration, $\theta_1\theta_2$, $\phi_1\phi_2$ the angles which the wave axes make with each of these lines; then

$$\frac{\cos\theta_1}{\cos\theta_2} = \frac{\alpha_1 \sqrt{(a^2 - b^2)} + \gamma_1 \sqrt{(b^2 - c^2)}}{\alpha \sqrt{(a^2 - b^2)} - \gamma_1 \sqrt{(b^2 - c^2)}} = \frac{A\dfrac{\alpha_1}{\gamma_1} + 1}{A\dfrac{\alpha_1}{\gamma_1} - 1},$$

$$\text{where} \quad A = \sqrt{\left(\frac{a^2 - b^2}{b^2 - c^2}\right)}.$$

$$\text{Similarly} \quad \frac{\cos\phi_1}{\cos\phi_2} = \frac{A\dfrac{\alpha_2}{\gamma_2} + 1}{A\dfrac{\alpha_2}{\gamma_2} - 1},$$

$$\therefore \frac{\cos\theta_1}{\cos\theta_2} \cdot \frac{\cos\phi_2}{\cos\phi_1} = \frac{A^2 \frac{\alpha_1\alpha_2}{\gamma_1\gamma_2} + A\left(\frac{\alpha_2}{\gamma_2} - \frac{\alpha_1}{\gamma_1}\right) - 1}{A^2 \frac{\alpha_1\alpha_2}{\gamma_1\gamma_2} - A\left(\frac{\alpha_2}{\gamma_2} - \frac{\alpha_1}{\gamma_1}\right) - 1}.$$

Now it is shewn (*Mathematical Journal*, vol. I. p. 79), that

$$\frac{\alpha_1\alpha_2}{\gamma_1\gamma_2} = \frac{1}{A^2};$$

$$\text{hence} \quad \frac{\cos\theta_1}{\cos\theta_2} = -\frac{\cos\phi_1}{\cos\phi_2};$$

which shows that the projections of the wave axes on the plane of the wave make equal angles with the directions of rectilinear vibration. For let χ, ψ be the angles the wave axes make respectively with the normal to the front, ω, ω' the angles their projections on the front make with one of the directions of rectilineal vibration, then it is easily seen that

$$\cos\theta_1 = \sin\chi\cos\omega, \quad \cos\phi_1 = \sin\psi\cos\omega',$$

$$\cos\theta_2 = \sin\chi\sin\omega, \quad \cos\phi_2 = \sin\psi\sin\omega',$$

and therefore the relation just proved becomes

$$\cot\omega = -\cot\omega', \quad \text{or} \quad \omega = -\omega',$$

since ω and ω' may be considered acute angles. And from hence the truth of the proposition is evident.

This proof appeared under a more complicated form in *Liouville's Mathematical Journal*, Sept. 1843.

VI.—ON A MULTIPLE DEFINITE INTEGRAL.

By R. L. Ellis, M.A. Fellow of Trinity College.

In the eighteenth number of the Journal, I pointed out the mode in which Fourier's theorem may be employed in the evaluation of certain definite multiple integrals. The theorem generally known as Liouville's, and another of the same degree of generality, were readily deduced from the considerations then suggested. I proceed to another application of the same method.

Theor. $$\int dx \int dy \ldots \frac{f(mx + ny + \ldots)}{(a^2 + x^2)(b^2 + y^2)\ldots}$$

$$= \pi^{\nu-1} \frac{ma + nb + \ldots}{ab \ldots} \int_h^{h'} \frac{fu\, du}{u^2 + (ma + nb + \ldots)^2};$$

the integral being supposed to involve ν variables x, y, &c., and the limits being given by the inequalities

$$mx + ny + \ldots \geqq h \text{ and } \leqq h'.$$

(Negative as well as positive values of the variables are admissible.)

DEM. Recurring to the general theorem stated at the commencement of the paper already mentioned, we see that the integral whose value is sought is equal to

$$\frac{1}{\pi}\int_h^{h'} fu\,du \int_0^\infty da \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \ldots \frac{\cos a\,(mx + ny + \ldots - u)}{(a^2 + x^2)(b^2 + y^2)\ldots}.$$

Develope the cosine in a series of products of sines and cosines of simple arcs. Every term involving a sine disappears on integration, as the limits extend from $-\infty$ to $+\infty$. Consequently the last written expression becomes

$$\frac{1}{\pi}\int_h^{h'} fu\,du \int_0^\infty \cos au\,da \int_{-\infty}^{+\infty} \frac{\cos amx}{a^2 + x^2}\,dx \int_{-\infty}^{+\infty} \frac{\cos any}{b^2 + y^2}\,dy \ldots.$$

Now $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos amx}{a^2 + x^2}\,du = \frac{\pi}{a}\,e^{-ma\alpha}$ &c. = &c.,

and thus the integral becomes

$$\frac{\pi^{\nu-1}}{ab\ldots}\int_h^{h'} fudu \int_0^\infty \cos aue^{-(ma+nb+\ldots)\alpha}\,da,$$

$$\text{or,}\quad \pi^{\nu-1}\,\frac{ma + nb + \ldots}{ab\ldots}\int_h^{h'} \frac{fu\,du}{u^2 + (ma + nb + \ldots)^2};$$

which was to be proved.

It may be well to verify this result in a particular case. Let $\nu = 2$; then we have to prove that

$$\int dx \int dy\,\frac{f(mx + ny)}{(a^2 + x^2)(b^2 + y^2)} = \pi\,\frac{ma + nb}{ab}\int_h^{h'} \frac{fu\,du}{u^2 + (ma + nb)^2};$$

for simplicity, we will suppose that

$$m^2 + n^2 = 1.$$

Let $\quad mx + ny = u,\quad$ and therefore $\quad x = mu + nv,$

$\qquad nx - my = v, \qquad\qquad\qquad y = nu - mv,$

u and v being two new variables. As $u^2 + v^2 = x^2 + y^2$, $dxdy$ is to be replaced by $dudv$, and thus

$$\int dx \int dy\,\frac{f(mx + ny)}{(a^2 + x^2)(b^2 + y^2)} \text{ becomes}$$

$$\int fudu \int \frac{dv}{\{a^2 + (mu + nv)^2\}\{b^2 + (nu - mv)^2\}}.$$

The limits are easily seen to be $+\infty$ $-\infty$ for v; h' and h for u. For the integral expresses the volume of that portion of a solid bounded by the surface, whose equation is

$$z = \frac{f(mx + ny)}{(a^2 + x^2)(b^2 + y^2)};$$

which is included between the plane of (xy), the bounding surface, and two planes parallel to one another and perpendicular to (xy). The equations of these planes are respectively

$$mx + ny = h, \text{ and } mx + ny = h',$$

and the introduction of u and v is equivalent to changing the axes of co-ordinates, so that one of the new axes, that of u, is perpendicular to these planes, while the other is parallel to them.

In order to find the value of

$$\int_{-\infty}^{+\infty} \frac{dv}{\{a^2 + (mu + nv)^2\}\{b^2 + (nu - mv)^2\}}, \text{ assume}$$

$$\frac{1}{\{a^2 + (mu + nv)^2\}\{b^2 + (nu - mv)^2\}}$$

$$= \frac{A(mu + nv) + Bna}{a^2 + (mu + nv)^2} + \frac{C(nu - mv) + Dmb}{b^2 + (nu - mv)^2}.$$

It is evident that the terms in A and C will disappear on integration between infinite limits: those in B and D become respectively πB and πD, and the integral in question is therefore

$$\pi(B + D).$$

Now it may be shown that

$$B = \frac{n}{a} \frac{u^2 - m^2a^2 + n^2b^2}{\{u^2 + (ma + nb)^2\}\{u^2 + (ma - nb)^2\}};$$

$$D = \frac{m}{b} \frac{u^2 + m^2a^2 - n^2b^2}{\{u^2 + (ma + nb)^2\}\{u^2 + (ma - nb)^2\}}.$$

Consequently $B + D = \dfrac{ma + nb}{ab} \dfrac{1}{u^2 + (ma + nb)^2}$;

and thus the integral sought is seen to be equal to

$$\pi \frac{ma + nb}{av} \int_h^{h'} \frac{fudu}{u^2 + (ma + nb)^2},$$

which was to be proved.

Similar considerations apply in the case of more variables, and doubtless by induction our general result might be established. But the method we have followed, besides being more analytical, is also very much simpler.

Another result of this same kind of analysis, I shall indicate without a demonstration, which there will be no difficulty in supplying.

$$\int dx \int dy \ldots e^{-a^2x^2-b^2y^2\ldots} f(max + nby + \ldots)$$

$$= \frac{\pi^{\frac{n-1}{2}}}{ab\ldots} \frac{1}{\{m^2 + n^2 + \ldots\}^{\frac{1}{2}}} \int_h^{h'} e^{-\frac{u^2}{m^2+n^2+\ldots}} fudu,$$

the limits being given by

$$max + nby + \ldots \overset{=}{>} h \text{ and } \underset{=}{<} h'.$$

In conclusion, it may be well to remark, that the analysis of which we have made use is not unfrequently applicable to questions which though not difficult in principle are nevertheless somewhat perplexing in practice.

VII.—CHAPTERS IN THE ANALYTICAL GEOMETRY OF (n) DIMENSIONS.

By A. Cayley, B.A. Fellow of Trinity College.

Chap. 1. *On some preliminary formulæ.*

I take for granted all the ordinary formulæ relating to determinants. It will be convenient, however, to write down a few, relating to a certain system of determinants, which are of considerable importance in that which follows: they are all of them either known, or immediately deducible from known formulæ.

Consider the series of terms

$$\begin{array}{l} x_1,\ x_2 \ldots x_n \ \ldots\ldots\ldots\ldots\ldots (1), \\ A_1, A_2 \ldots A_n \\ \vdots \\ K_1, K_2 \ldots K_n \end{array}$$

the number of the quantities $A \ldots K_1$ being equal to r $(r < n)$. Suppose $(r + 1)$ vertical rows selected, and the quantities contained in them formed into a determinant, this may be done in $\dfrac{n.(n-1)\ldots(q+2)}{1.2\ldots n-q-1}$ different ways. The

system of determinants so obtained will be represented by the notation

$$\left\| \begin{matrix} x_1, & x_2 \dots & x_n \\ A_1, & A_2 \dots & A_n \\ \vdots & & \vdots \\ K_1, & K_2 \dots & K_n \end{matrix} \right\| \dots\dots\dots\dots (2).$$

And the system of equations, obtained by equating each of these determinants to zero, by the notation

$$\left\| \begin{matrix} x_1, & x_2 \dots & x_n \\ A_1, & A_2 \dots & A_n \\ \vdots & & \vdots \\ K_1, & K_2 \dots & K_n \end{matrix} \right\| = 0 \dots\dots\dots (3).$$

The $\dfrac{n.(n-1)\dots(q+2)}{1.2\dots(n-q+1)}$ equations represented by this formula reduce themselves to $(n-q)$ independent equations. Imagine these expressed by

$$(1) = 0, \quad (2) = 0 \dots\dots (n-q) = 0 \dots\dots (4).$$

Any one of the determinants of (2) is reducible to the form

$$\Theta_1 (1) + \Theta_2 (2) \dots + \Theta_{n-q} (n-q) \dots\dots\dots (5),$$

where $\Theta_1, \Theta_2 \dots \Theta_{n-q}$ are coefficients independent of $x_1, x_2 \dots x_n$. The equations (3) may be replaced by

$$\left\| \begin{matrix} \lambda_1 x_1 + \lambda_2 x_2 + \dots \lambda_n x_n, & \mu_1 x_1 + \dots, & \dots \tau_1 x_1 + \dots \\ \lambda_1 A_1 + \lambda_2 A_2 + \dots \lambda_n x_n, & \mu_1 A_1 + \dots, & \dots \tau_1 A_1 + \dots \\ \vdots & & \\ \lambda_1 K_1 + \lambda_2 K_2 + \dots \lambda_n K_n, & \mu_1 K_1 + \dots, & \tau_1 K_1 + \dots \end{matrix} \right\| = 0 \dots (6).$$

And conversely from (6) we may deduce (3), unless

$$\left| \begin{matrix} \lambda_1, & \lambda_2, \dots & \lambda_n \\ \mu_1, & \mu_2 \dots & \mu_n \\ \vdots & & \\ \tau_1, & \tau_2 \dots & \tau_n \end{matrix} \right| = 0 \dots\dots\dots\dots (7).$$

(The number of the quantities $\lambda, \mu \dots \tau$ is of course equal to n). The equations (3) may also be expressed in the form

$$\left\| \begin{matrix} x_1, & x_2, & \dots & x_n \\ \lambda_1 A_1 + \dots \omega_1 K_1, & \lambda_1 A_2 + \dots \omega_1 K_2, & \dots & \lambda_1 A_n \dots + \omega_1 K_n \\ \vdots & \vdots & & \\ \lambda_r A_1 + \dots \omega_r K_1, & \lambda_r A_2 + \dots \omega_r K_2, & \dots & \lambda_r A_n \dots + \omega_r K_n \end{matrix} \right\| \dots (8).$$

The number of the quantities $\lambda, \mu \dots \omega$ being (r).

And conversely (3) is deducible from (8), unless

$$\begin{vmatrix} \lambda_1 \dots \omega_1 \\ \vdots \\ \lambda_r \dots \omega_r \end{vmatrix} = 0 \quad \dots\dots\dots\dots (9).$$

CHAP. 2. *On the determination of linear equations in $x_1, x_2 \dots x_n$ which are satisfied by the values of these quantities derived from given systems of linear equations.*

It is required to find linear equations in $x_1 \dots x_n$ which are satisfied by the values of these quantities derived— 1. from the equations $\mathfrak{A}' = 0$, $\mathfrak{B}' = 0 \dots \mathfrak{G}' = 0$; 2. from the equations $\mathfrak{A}'' = 0$, $\mathfrak{B}'' = 0 \dots \mathfrak{O}'' = 0$; 3. from $\mathfrak{A}''' = 0$, $\mathfrak{B}''' = 0 \dots \mathfrak{R}''' = 0$, &c. &c., where

$$\begin{aligned} \mathfrak{A} &= A_1 x_1 + A_2 x_2 \dots + A_n x_n \dots\dots\dots\dots (1). \\ \mathfrak{B} &= B_1 x_1 + B_2 x_2 \dots + B_n x_n . \\ &\vdots \end{aligned}$$

Also $r', r'' \dots$ representing the number of equations in the systems (1), (2)... and k the number of these given systems, $(n - r') + (n - r'') + \dots \not> n - 1$ or $(k - 1)\,n + 1 \not> r' + r'' + \dots$

Assume
$$\begin{aligned} C\Psi &= \lambda'\mathfrak{A}' + \mu'\mathfrak{B}' + \dots\dots\dots\dots (2). \\ &\lambda'\mathfrak{A}' + \mu'\mathfrak{B}' + \dots = \\ &\lambda''\mathfrak{A}'' + \mu''\mathfrak{B}'' + \dots = \\ &\lambda'''\mathfrak{A}''' + \mu'''\mathfrak{B}''' + \dots = \\ &\text{\&c.} \end{aligned}$$

The latter equations being satisfied for the terms involving x_1, for those involving x_2, &c...separately. Suppose, in addition to these, a set of linear equations in $\lambda', \mu' \dots \lambda'' \mu'' \dots$ so that, with the preceding ones, there is a sufficient number of equations for the elimination of these quantities. Then, performing the elimination, the value of $\mathfrak{N}$, so obtained, is a function of $x_1, x_2 \dots$ which vanishes for the values of these quantities, derived from the equations (1) or (2)...&c. The series of equations $\Psi = 0$ may be expressed in the form

$$\left\| \begin{array}{llllll} \mathfrak{A}' \; \mathfrak{B}' \dots \mathfrak{C}' & & & & & \\ A'_1 B'_1 \dots G'_1 & A''_1 \dots O''_1 & & & & \\ \vdots \;\; \vdots \quad\; \vdots & \vdots \quad\;\; \vdots & & & & \\ A'_n B'_n \dots G'_n & A''_n \dots O''_n & & & & \\ & A''_1 \dots O''_1 & A'''_1 \dots R'''_1 & & & \\ & \vdots \quad\;\; \vdots & \vdots \quad\;\; \vdots & & & \\ & A''_n \dots O''_n & A'''_n \dots R'''_n & & & \\ & & . \quad\;\; . & . & . & \\ & & . \quad\;\; . & . & . & \end{array} \right\| = 0 \dots\dots (3).$$

Chap. 3. *On reciprocal equations.*

Consider a system of equations

$$\begin{aligned} &A_1x_1 + A_2x_2 \dots + A_nx_n = 0 \quad \dots\dots\dots\dots (1),\\ &\quad\vdots\\ &K_1x_1 + K_2x_2 \dots + K_nx_n = 0 \end{aligned}$$

(r in number).

The reciprocal system with respect to a given function (U) of the second order in $x_1, x_2 \dots x_n$, is said to be

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \dots d_{x_n}U \\ A_1, & A_2 \dots\dots A_n \\ \vdots & \qquad\quad \vdots \\ K_1, & K_2 \dots\dots K_n \end{matrix} \right\| = 0 \quad \dots\dots\dots (2),$$

($n - r$ in number).

It must first be shewn that the reciprocal system to (2) is the system (1), or that the systems (1), (2) are reciprocals of each other.

Consider, in general, the system of equations

$$\begin{aligned} &a_1d_{x_1}U + a_2d_{x_2}U \dots + a_nd_{x_n}U = 0 \quad \dots\dots\dots (3).\\ &\quad\vdots\\ &\lambda_1d_{x_1}U + \lambda_2d_{x_2}U \dots + \lambda_nd_{x_n}U = 0. \end{aligned}$$

Suppose $\qquad 2U = \Sigma(a^2)\,x_a^{\,2} + 2\Sigma(a\beta)\,x_ax_b,$

so that $\quad d_{x_s}U = \Sigma(sa)\,x_a \dots\dots\dots\dots (4), (5).$

The equations (3) may be written

$$\begin{aligned} &x_1\{a_1(1^2) + a_2(12) \dots + a_n(1n)\} + \dots\\ &\text{\&c.} \qquad\qquad + x_n\{a_1(n1) + a_2.(n2) \dots + a_n(n^2)\} = 0 \dots (6). \end{aligned}$$

And forming the reciprocals of these, also replacing $d_{x_1}U$, $d_{x_2}U \dots$ by their values, we have

$$\left\| \begin{matrix} x_1(1^2) + x_2(12) + \dots x_n(1n) \dots x_1(n1) + x_2(n.2) \dots + x_n(n^2) \\ a_1(1^2) + a_2(12) + \dots a_n(1.n) \dots a_1(n1) + a_2(n.2) \dots + a_n(n^2) \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \lambda_1(1^2) + \lambda_2(12) + \dots \lambda_n(1.n) \dots \lambda_1(n1) + \lambda_2(n2) \dots + \lambda_n(n^2) \end{matrix} \right\| = 0 \dots (7).$$

From which, assuming

$$\left\| \begin{matrix} (1^2)\,(12) \dots (1n) \\ (21).(2^2) \dots (2n) \\ \vdots \\ (n1)\,(n2) \dots (n^2) \end{matrix} \right\| = 0 \quad \dots\dots\dots\dots (8).$$

We have, for the reciprocal system of (3),

$$\left\| \begin{matrix} x_1, & x_2 \ldots x_n \\ a_1, & a_2 \ldots a_n \\ \vdots & \\ \lambda_1, & \lambda_2 \ldots \lambda_n \end{matrix} \right\| = 0 \ldots\ldots\ldots\ldots (9).$$

Now, suppose the equations (3) represent the system (2); their number in this case must be $(n - r)$. Also if θ represent any one of the quantities $a, \beta \ldots \lambda$, we have

$$\begin{aligned} &A_1\theta_1 + A_2\theta_2 \ldots + A_n\theta_n = 0 \ldots\ldots\ldots\ldots (10). \\ &\vdots \\ &K_1\theta_1 + K_2\theta_2 \ldots + K_n\theta_n = 0. \end{aligned}$$

By means of these equations, the system (9) may be reduced to the form

$$\left\| \begin{matrix} A_1x_1+A_2x_2\ldots+A_nx_n & \ldots & K_1x_1+K_2x_2\ldots+K_nx_n. & x_{r+1}, x_{r+2} \ldots x_n \\ 0 & \ldots & 0 & a_{r+1}, a_{r+2} \ldots a_n \\ & & & \vdots \\ 0 & \ldots & 0 & \lambda_{r+1}, \lambda_{r+2} \ldots \lambda_n \end{matrix} \right\| = \ldots (11),$$

which are satisfied by the equations (1). Hence the reciprocal system to (2) is (1), or (1), (2) are reciprocals to each other.

THEOREM. Consider the equations

$$\begin{aligned} &(\mathfrak{A}' = 0, \ \mathfrak{B}' = 0 \ldots \mathfrak{G}' = 0) \ldots\ldots\ldots (12), \\ &(\mathfrak{A}'' = 0, \ \mathfrak{B}'' = 0 \ldots \mathfrak{N}'' = 0), \\ &(\mathfrak{A}''' = 0, \ \mathfrak{B}''' = 0 \ldots \mathfrak{K}''' = 0), \\ &\&c. \end{aligned}$$

of Chap. 2. The equations

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots & d_{x_n}U \\ A'_1, & A'_2 \ldots & A'_n \\ \vdots & & \vdots \\ G'_1, & G'_2 \ldots & G'_n \end{matrix} \right\| = 0 \ldots\ldots\ldots (13),$$

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots & d_{x_n}U \\ A''_1, & A''_2 \ldots & A''_n \\ \vdots & & \\ O''_1, & O''_2 \ldots & O''_n \end{matrix} \right\| = 0,$$

&c.

which are the reciprocals of these systems, represent taken conjointly, the reciprocal of the system of equations (3) of the same chapter.

Let this system, which contains $n - \{(n-r) + (n-r') + \ldots\}$ equations, be represented by

$$\begin{aligned} &a_1x_1 + a_2x_2 \cdots + a_nx_n = 0 \ldots\ldots\ldots\ldots (14).\\ &\beta_1x_1 + \beta_2x_2 \cdots + \beta_nx_n = 0.\\ &\vdots\\ &\zeta_1x_1 + \zeta_2x_2 \cdots + \zeta_nx_n = 0. \end{aligned}$$

The reciprocal system is

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots d_{x_n}U \\ a_1, & a_2 \ldots a_n \\ \vdots & \\ \zeta_1, & \zeta_2 \ldots \zeta_n \end{matrix} \right\| = 0 \ldots\ldots\ldots (15).$$

containing $(n-r) + (n-r') +$ &c. ... equations.

Also, by the formulæ in Chap. 2,

$$\begin{aligned} &a_1x_1 + \ldots + a_nx_n = \lambda'_1\mathfrak{A}' + \mu'_1\mathfrak{B}' + \ldots \sigma'_1\mathfrak{G}' \quad (\lambda, \mu \ldots \sigma, r' \text{ in number}).\\ &\beta_1x_1 + \ldots + \beta_nx_n = \lambda'_2\mathfrak{A}' + \mu'_2\mathfrak{B}' + \ldots \sigma'_2\mathfrak{G}'\\ &\vdots\\ &\zeta_1x_1 \quad \ldots + \zeta_nx_n = \lambda_\theta'\mathfrak{A}' + \mu_\theta'\mathfrak{B}' + \ldots \sigma_\theta'\mathfrak{G}' \ldots\ldots\ldots\ldots\ldots (16), \end{aligned}$$

writing $\theta = n - \{(n-r) + (n-r') + \ldots\}$.

Also, assuming any arbitrary quantities $\eta_1, \eta_2 \ldots \eta_n \ldots \phi_1, \phi_2 \ldots \phi_n$ (the number of sets being $(r' - \theta)$,) such that

$$\begin{aligned} &\eta_1x_1 \ldots + \eta_nx_n = \lambda'_{\theta+1}\mathfrak{A}' + \mu'_{\theta+1}\mathfrak{B}' + \ldots \sigma_{\theta+1}\mathfrak{G}' \ldots (17).\\ &\phi_1x_1 \ldots + \phi_nx_n = \lambda'_{r'}\mathfrak{A}' + \mu'_{r'}\mathfrak{B}' + \ldots \sigma'_{r'}\mathfrak{G}'. \end{aligned}$$

From the equations (15) we deduce the $(n-r)$ equations

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots d_{x_n}U \\ a_1, & a_2 \ldots a_n \\ \vdots & \\ \phi_1, & \phi_2 \ldots \phi_n \end{matrix} \right\| = 0 \ldots\ldots\ldots (18).$$

Hence, writing

$$\begin{aligned} &a = \lambda_1'A + \mu_1'B + \ldots \sigma_1'G \ldots\ldots\ldots\ldots (19),\\ &\beta = \lambda_2'A + \mu_2'B + \ldots \sigma_2'G,\\ &\vdots\\ &\phi = \lambda_r'A + \mu_r'B + \ldots \sigma_r'G. \end{aligned}$$

And reducing, by the formula (8) of Chap. 1, we have

$$\left\| \begin{matrix} d_{x_1}U, & d_{x_2}U \ldots d_{x_n}U \\ A_1', & A_2' \ldots A_n' \\ \vdots & \\ G_1', & G_2' \ldots G_n' \end{matrix} \right\| = 0 \ldots\ldots\ldots (20).$$

And similarly may the remaining formulæ of (13) be deduced

from the equation (15). Hence the required theorem is demonstrated, a theorem which may be more clearly stated as follows:—

The reciprocals of several systems of equations form together the reciprocal of the equation which is satisfied by the values of the variables which satisfy each of the original systems of equations. (The theorem requires that the number of all the reciprocal equations shall be less than the number of variables.)

Conversely, consider several systems of equations, the whole number of the equations being less than the number of variables. These systems, taken conjointly, have for their reciprocal, the equation which is satisfied by the values satisfying the reciprocal system of each of the given systems.

CHAP. 4. *On some properties of functions of the second order.*

Suppose, as before, U denotes the general function of the second order, or

$$2U = \Sigma\,(a^2)\,.\,x_a^2 + 2\Sigma\,(a\beta)\,x_a x_\beta \;\ldots\ldots\; (21).$$

Also let V denote a function of the second order of the form

$$V = H\left(\left\|\begin{matrix} x_1, & x_2 \ldots x_n \\ a_1, & a_2 \ldots a_n \\ \vdots & \\ x_1, & x_2 \ldots x_n \end{matrix}\right\|\right) \;\ldots\ldots\ldots\; (22),$$

(H being the symbol of a homogeneous function of the second order, and the number r of the quantities $a, \beta \ldots x$, being less than $(n-1)$.) Then $2U - 2kV$, k arbitrary, is of the form

$$\Sigma\,[a^2]\,x_a^2 + 2\Sigma\,[a\beta]\,x_a x_\beta \;\ldots\ldots\ldots\; (23).$$

Suppose $X_1, X_2 \ldots X_n$ determined by the equations

$$[1^2]\,X_1 + [12]\,X_2 \ldots + [1n]\,X_n = 0 \;\ldots\ldots (24),$$
$$[21]\,X_1 + [2^2]\,X_2 \ldots + [2n]\,X_n = 0,$$
$$\vdots$$
$$[n1]\,X_1 + [n2]\,X_2 \ldots + [n^2]\,X_n = 0\,;$$

equations that involve the condition that k satisfies a certain of the order $(n-r)$, as will be presently proved.

Then shall $x_1 = X_1 \ldots x_n = X_n$ satisfy the system of equations, which is the reciprocal of

$$\left\|\begin{matrix} x_1, & x_2 \ldots x_n \\ a_1, & a_2 \ldots a_n \\ \vdots & \\ x_1, & x_2 \ldots x_n \end{matrix}\right\| = 0 \;\ldots\ldots\ldots\ldots\; (25).$$

To prove these properties, in the first place, we must find the form of V. Consider the quantities $\xi_A, \xi_B \dots \xi_L$, $(n-r)$ in number, of the form

$$\xi_A = A_1 x_1 + A_2 x_2 \dots + A_n x_n \dots\dots\dots\dots (26),$$
$$\xi_B = B_1 x_1 + B_2 x_2 \dots + B_n x_n,$$
$$\vdots$$
$$\xi_L = L_1 x_1 + L_2 x_2 \dots + L_n x_n;$$

where, if Θ represent any of the quantities $A, B \dots L$,

$$\alpha_1 \Theta_1 + \alpha_2 \Theta_2 \dots + \alpha_n \Theta_n = 0 \dots\dots\dots\dots (27),$$
$$\beta_1 \Theta_1 + \beta_2 \Theta_2 \dots + \beta_n \Theta_n = 0,$$
$$\vdots$$
$$x_1 \Theta_1 + x_2 \Theta_2 \dots + x_n \Theta_n = 0.$$
$$2V = (A^2)\,\xi_A^2 + (B^2)\,\xi_B^2 + \dots$$
$$+ 2\,(AB)\,\xi_A \xi_B + \dots$$
$$= \Sigma\,(A^2)\,\xi_A^2 + 2\Sigma\,(AB)\,\xi_A \xi_B\,.$$

Hence, if $\quad 2V = \Sigma\,\{a^2\}\,x_a^2 + 2\Sigma\,\{\alpha\beta\}\,\xi_A \xi_B \dots\dots\dots (28),$

$$\{1^2\} = \Sigma(A^2)\,A_1^2 + 2\Sigma(AB)\,A_1 B_1,$$
$$\vdots$$
$$\{12\} = \Sigma(A^2)\,A_1 A_2 + \Sigma\,(AB)\,(A_1 B_2 + A_2 B_1),$$
$$\vdots$$
$$[1^2] = (1^2) - k\,\{1^2\},$$
$$\vdots$$
$$[12] = (12) - k\,\{12\}.$$
$$\vdots$$

Hence, θ representing any of the quantities $\alpha, \beta \dots x$,

$$\theta_1\,\{1^2\} + \theta_2\,\{12\} \dots + \theta_n\,\{1n\} = 0 \dots\dots (29),$$
$$\vdots$$
$$\theta_1\,\{n1\} + \theta_2\,\{n2\} \dots + \theta_n\,\{n^2\} = 0;$$

whence also

$$\theta_1\,[1^2] + \dots \theta_n\,[1n] = \theta_1\,(1^2) + \dots \theta_n\,(1n),$$
$$\vdots$$
$$\theta_n\,[n1] + \dots \theta_n\,[n^2] = \theta_1\,(n1) + \dots\ \theta_n\,(n^2).$$

Hence, the equations for determining $X_1 \dots X_n$ may be reduced to

$$X_1[\alpha_1(1^2) + \dots \alpha_n(1n)] + X_2[\alpha_1(21) \dots + \alpha_n(2n)] \dots + X_n[\alpha_1(n1) \dots + \alpha_n(n^2)] = 0 \dots (30),$$
$$X_1[\beta_1(1^2) + \dots \beta_n(1n)] + X_2[\beta_1(21) \dots + \beta_n(2n)] \dots + X_n[\beta_1(n1) \dots + \beta_n(n^2)] = 0,$$
$$\vdots$$
$$X_1[n_1(1^2) + \dots n_n(1n)] + X_2[n_1(21) \dots + n_n(2n)] \dots + X_n[n_n(n1) \dots + n_n(n^2)] = 0.$$
$$X_1[r+1, 1] + X_2\,[r+1, 2] \dots + X_n\,[r+1, n] \qquad = 0,$$
$$\vdots$$
$$X_1[n, 1] + X_2\,[n, 2] \dots + X_n\,[n^2] \qquad = 0.$$

Eliminating $X_1 \ldots X_n$, since the first (r) equations do not contain k, the equation in this quantity is of the order $(n-r)$.

Next form the reciprocals of the equations (25). These are

$$\left\| \begin{matrix} d_{x_1} U, & d_{x_2} U \ldots d_{x_n} U \\ A_1, & A_2 \ldots A_n \\ \vdots & \\ L_1, & L_2 \ldots L_n \end{matrix} \right\| = 0 \quad \ldots\ldots\ldots \text{(31)}.$$

From which we may deduce

$$\left\| \begin{matrix} d_{x_1} U \ldots + a_n d_{x_n} U, & \beta_1 d_{x_1} U \ldots + \beta_n d_x U \ldots & x_1 d_{x_1} U \ldots x_n d_{x_n} U, & d_{x_{r+1}} U \ldots d_{x_n} U \\ 0 \quad , & 0 \quad \ldots & 0 & A_{r+1} \ldots A_n \\ \vdots & & & \\ 0 \quad , & 0 \quad \ldots & 0 & L_{r+1} \ldots L_n \end{matrix} \right\| = 0 \ldots \text{(32)}.$$

which is evidently satisfied by $x_1 = X_1$, $x_2 = X_2 \ldots x_n = X_n$.

In the case of four variables, the above investigation demonstrates the following properties of surfaces of the second order.

I. If a cone intersect a surface of the second order, three different cones may be drawn through the curve of intersection, and the vertices of these lie in the plane which is the polar reciprocal of the vertex of the intersecting cone.

II. If two planes intersect a surface of the second order through the curve of intersection, two cones may be drawn, and the vertices of these lie in the line which is the polar reciprocal of the line of intersection of the two planes.

Both these theorems are undoubtedly known, though I am not able to refer for them to any given place.

VIII.—ON A QUESTION IN THE THEORY OF PROBABILITIES.

By R. L. Ellis, M.A. Fellow of Trinity College.

The following question affords a good illustration of the methods employed in the more difficult parts of the theory of probabilities. In a paper presented to the Philosophical Society, I applied the kind of analysis we are about to make use of, to the celebrated *Rule of Least Squares*. There is, in fact, a close analogy between the two investigations. Laplace's solution of the present question is obtained by a process similar to that which he had employed when treating of the best method of combining discordant observations.

What is the probability that the sum of the times which each of n persons has respectively yet to live will amount to a given time T?

Let $\phi_p x_p dx_p$ be the probability that the p^{th} person will live precisely a time x_p longer, ϕ_p denoting some function of x_p, which is necessarily such that

$$\int_0^\infty \phi_p x_p dx_p = 1,$$

as it is certain that he will die at some time or other.

Let $x_1, x_2 \ldots x_n$ be so related that

$$x_1 + x_2 + \ldots x_n = T.$$

The probability of this particular combination is

$$\phi_1 x_1 \phi_2 x_2 \cdots \phi_n x_n dx_1 dx_2 \ldots dx_n,$$

or $$\phi_1 x_1 \phi_2 x_2 \ldots \phi_n (T - x_1 - \ldots x_{n-1})\, dx_1 \ldots dx_n,$$

and the aggregate probability sought is the integral of this expression obtained by giving all possible positive values to $x_1 \ldots x_{n-1}$, which do not make $T - x_1 - \ldots x_{n-1}$ negative. Thus we have

$$P = dx_n \textstyle\int_0 \ldots \int_0 \phi_1 x_1 \phi_2 x_2 \cdots \phi_n (T - x_1 - \ldots x_{n-1})\, dx_1 \ldots dx_{n-1}.$$

Now, by Fourier's theorem,

$$\phi_n (T - x_1 - \ldots x_{n-1}) = \frac{1}{\pi} \int_0^\infty d\alpha \int_0^\infty \phi_n x_n \cos \alpha (T - x_1 - \ldots x_{n-1} - x_n)\, dx_n,$$

$T - x_1 \ldots x_{n-1}$ being supposed to lie between 0 and ∞; for all negative values of this quantity, the second member of the equation is equal to zero. Consequently, all the integrations may now be taken from zero to infinity; and thus, since as T and x_n vary together, $dT = dx_n$

$$P = \frac{dT}{\pi} \int_0^\infty d\alpha \int_0^\infty dx_1 \ldots \int_0^\infty dx_n \phi_1 x_1 \ldots \phi_n x_n \cos \alpha\, (T - \Sigma x).$$

Now it may be shewn that the greatest value of

$$\int_0^\infty dx_1 \ldots \int_0^\infty dx_n \phi_1 x_1 \ldots \phi_n x_n \cos \alpha\, (T - \Sigma x) \ldots (a)$$

corresponds to $\alpha = 0$, and is, therefore, unity; and that when n is large (a) diminishes rapidly as α increases. Consequently the value of $\int_0^\infty (a)\, d\alpha$ depends, when n is very large, on the elements for which α is very small. This consideration enables us to employ an approximate value of (a).

Let $T = t + m$, m being a disposable quantity; then

$$(a) = \cos at \int_0^\infty dx_1 \dots \int_0^\infty dx_n \phi_1 x_1 \dots \phi_n x_n \cos a(m - \Sigma x)$$
$$+ \sin at \int_0^\infty dx_1 \dots \int_0^\infty dx_n \phi_1 x_1 \dots \phi_n x_n \sin a(m - \Sigma x),$$

which may be thus written

$$(a) = \cos atG + \sin atH.$$

In order to obtain approximate values of G and H, expand $\cos a(m - \Sigma x)$ and $\sin a(m - \Sigma x)$; they become respectively, a being very small,

$$1 - \tfrac{1}{2} a^2 (m - \Sigma x)^2 \text{ and } a(m - \Sigma x).$$

Now let $\int_0^\infty \phi x\, x dx = K \quad \int_0^\infty \phi x\, x^2 dx = k^2$;

then, since $\int_0^\infty \phi x dx = 1$, we shall have

$$G = 1 - \tfrac{1}{2} a^2 (m^2 - 2m\Sigma K + 2\Sigma K_1 K_2 + \Sigma k^2),$$
$$H = a(m - \Sigma K) \text{ approximately.}$$

Let $m = \Sigma K$, then

$$m^2 - 2m\Sigma K = -\{\Sigma K\}^2 = -\Sigma K^2 - 2\Sigma K_1 K_2,$$

and thus
$$G = 1 - \tfrac{1}{2} a^2 \Sigma (k^2 - K^2),$$
$$H = 0;$$

and therefore, while a is very small,

$$(a) = \cos at \{1 - \tfrac{1}{2} a^2 \Sigma (k^2 - K^2)\}.$$

We have next to show that $\Sigma (k^2 - K^2)$ is a positive quantity.

Consider the definite integral

$$\int_0^\infty \int_0^\infty \phi x \phi z (z - x)^2 dx dz;$$

it is necessarily positive, since every element is so, $\phi x dx$ being the expression of a probability, and therefore essentially positive.

Expanding $(z - x)^2$, we find for the value of this integral

$$k^2 - 2K^2 + k^2 \text{ or } 2(k^2 - K^2).$$

Hence $k^2 - K^2$, and therefore $\Sigma (k^2 - K^2)$, is positive.

(This demonstration is due to Poisson, *Con. des Tems*, 1827).

Returning to the value we have found for (a), we see that we may in all cases represent (a) by $\cos at\, e^{-\frac{1}{2} a^2 \Sigma (k^2 - K^2)}$, since when a is very small, the two expressions tend to coincide;

K

and when a is not so, both are sensibly zero, $\Sigma(k^2 - K^2)$ being a large quantity of the order n. Consequently

$$\int_0^\infty (a)\,da = \sqrt{\left(\frac{\pi}{2}\right)} \frac{1}{\{\Sigma(k^2 - K^2)\}^{\frac{1}{2}}} e^{-\frac{t^2}{2\Sigma(k^2-K^2)}}$$

$$\text{and}\quad P = \frac{dt}{\sqrt{(2\pi)}} \frac{1}{\{\Sigma(k^2 - K^2)\}^{\frac{1}{2}}} e^{-\frac{t^2}{2\Sigma(k^2-K^2)}} \ldots (p),$$

P being the probability that the required sum T shall be precisely equal to $\Sigma K + t$. The greatest value of P corresponds to $t = 0$; consequently the most probable value of T is ΣK.

It is to be remarked that the approximate formula (p) is independent of the law of probability expressed by the function ϕx: it depends merely on the two definite integrals

$$\int_0^\infty x\phi x dx \quad \text{and} \quad \int_0^\infty x^2 \phi x dx.$$

We have stopped the approximation to the value of G at the second power of a. Had we gone farther, and retained only the *principal* term in the coefficient of each power of a, a similar result, viz. one which may be assumed as coincident with the exponential function, would, there is little doubt, have been obtained; while the coefficient of each power of a in H would be negligible in comparison of the corresponding power in G. Some remarks on this, or at least on a cognate question, will be found in the paper already mentioned.

As a verification of the approximation we have employed, which is in effect the same as that of Laplace, let us suppose that the functions ϕ_1, $\phi_2 \ldots$ are all of the same form ϕ, and that $\phi x = e^{-x}$. Then, as we have seen, the required probability is obtained by integrating

$$e^{-x_1-x_2\ldots-(T-x_1-x_2-\ldots)}\,dx_1 \ldots dx_{n-1},$$

for all positive values of x_1, $x_2 \ldots x_{n-1}$ which do not transgress the limits

$$x_1 + x_2 \ldots x_{n-1} = T.$$

Now $$e^{-x_1-x_2\ldots-(T-x_1-x_2-\ldots)} = e^{-T},$$

and thus we have $P = e^{-T} dT \int_0 dx_1 \ldots \int_0 dx_{n-1}$,

the limits being given by

$$x_1 + x_2 \ldots x_{n-1} \leqq T.$$

Hence, it is easily seen that

$$P = \frac{e^{-T}}{\Gamma(n)} T^{n-1} dT.$$

In order to compare this with the approximate expression (p), I remark that $T = n - 1$ renders P a maximum; assume, therefore,

$$T = n - 1 + t,$$

$$\text{then} \quad P = \frac{e^{-(n-1)}(n-1)^{n-1}}{\Gamma(n)} \, e^{-t}\left(1 + \frac{t}{n-1}\right)^{n-1} at.$$

Now, by Stirling's theorem, or by that which Binet proposes to substitute for it, (vide *Journal de l'Ecole Polytechnique*, XVI. p. 226), we have, when n is very large,

$$\Gamma(n) - = \sqrt{(2\pi)}\; e^{-(n-1)}(n-1)^{n-\frac{1}{2}}.$$

Consequently

$$\frac{e^{-(n-1)}(n-1)^{n-1}}{\Gamma(n)} = \frac{1}{\sqrt{\{2\pi(n-1)\}}} = \frac{1}{\sqrt{(2\pi n)}} \cdots q.p.$$

$$\text{Again} \left\{1 + \frac{t}{n-1}\right\}^{n-1} = 1 + t + 1.\left(1 - \frac{1}{n-1}\right)\frac{t^2}{1.2}$$

$$+ \left(1 - \frac{1}{n-1}\right)\left(1 - \frac{2}{n-1}\right)\frac{t^3}{1.2.3} + \&c.$$

$$= e^t + ft,$$

ft being a certain function of t and n. Therefore

$$e^{-t}\left(1 + \frac{t}{n-1}\right)^{n-1} = 1 + e^{-t} ft.$$

Now the coefficient of t^2 in ft is

$$-\frac{1}{(n-1).1.2},$$

$$\text{that of } t^3 \text{ is } \left\{-\frac{3}{n-1} + \frac{2}{(n-1)^2}\right\}\frac{1}{1.2.3},$$

and that of t^4 is

$$\left\{-\frac{6}{n-1} + \frac{11}{(n-1)^2} - \frac{6}{(n-1)^3}\right\}\frac{1}{1.2.3.4}.$$

Hence the coefficient of t^2 in $e^{-t} ft$ is

$$-\frac{1}{2(n-1)},$$

that of t^3 is

$$\frac{1}{2(n-1)} - \frac{1}{2(n-1)} + \frac{1}{3(n-1)^2} \quad \text{or} \quad \frac{1}{3(n-1)^2};$$

and, lastly, that of t^4 is

$$-\frac{1}{4(n-1)}+\frac{2}{4(n-1)}-\frac{2}{1.2.3(n-1)^2}-\frac{1}{4(n-1)}$$
$$+\frac{11}{1.2.3.4}\frac{1}{(n-1)^2}-\frac{1}{4(n-1)^3},$$

$$\text{or } \frac{1}{1.2}\left\{\frac{1}{2(n-1)}\right\}^2-\frac{1}{4(n-1)^3}.$$

Now if, in forming the approximate expression, we reject all terms of the form $\left\{\frac{t}{\sqrt{(n)}}\right\}^p \frac{1}{n^q}$, where q is different from zero, *i.e.* if we look on $\frac{t}{\sqrt{(n)}}$ as a quantity all whose powers are to be retained, except when divided by any power of n, the value of $e^{-t}\left(1+\frac{t}{n-1}\right)^{n-1}$ may be taken as equal to

$$1-\frac{t^2}{2(n-1)}+\frac{1}{1.2}\cdot\frac{t^4}{4(n-1)^2}-\&c.$$

which, as similar results would have been obtained had we pursued the investigation farther, might be shown to be equal to

$$e^{-\frac{t^2}{2(n-1)}},$$

and thus

$$P=\frac{1}{\sqrt{(2\pi n)}}\,e^{-\frac{t^2}{2(n-1)}}\,dt,$$

P is the probability that T is equal to $n-1+t$: writing $t+1$ for t in $\frac{t^2}{n-1}$ and reducing, we find that, within the limits of the approximation, it may also be assumed as the probability that t is equal to $n+t$: also $\frac{t^2}{n-1}=\frac{t^2}{n}$ .... *q.p.*, and thus

$$P=\frac{1}{\sqrt{(2\pi n)}}\,e^{-\frac{t^2}{2n}}\,dt;$$

which, as in our case, $k^2=2$ and $K=1$ is precisely equivalent to the result deduced from the general formula (p).

The legitimacy of some parts of the preceding approximation may be questioned; as quantities which are neglected may, under certain conditions, be larger than those which are retained: and, as the result coincides with that of the general method, the doubt thus suggested appears to extend to the latter. The subject of approximation by means of definite integrals is certainly not free from obscurity.

The method of this paper extends *m.m.* to the case in which we seek to determine the degree of improbability that the average length of the reigns of a series of kings shall exceed by a given quantity the average deduced from authentic history. The application of considerations of this nature to historical criticism appears to have been first made in Sir Isaac Newton's Chronology. They are doubtless entitled to much attention; but any attempt to evaluate their legitimate influence, would, for more than one reason, be unsatisfactory.

IX.—ON THE BALANCE OF THE CHRONOMETER.

It is well known that a common watch goes more slowly when its temperature is raised, and *versâ vice.* The reason of this is that the elasticity of the balance-spring decreases with the increment of temperature and increases with its decrement. Neglecting the mass of the spring and the connection of the balance with the other parts of the watch, we may take as the equation for determining the oscillations of the balance,

$$\frac{d^2\theta}{dt^2} + \frac{e\theta}{I} = 0,$$

where e depends on the form and elasticity of the spring, and I is the moment of inertia of the balance. The time of oscillation depends, of course, on the ratio $\frac{e}{I}$, e being, as we have said, a function of t the temperature. In order, therefore, to the equable rate of the watch, it would be necessary that I should be such a function of t, that $\frac{e}{I}$ may be constant. In the balance of a common watch I is sensibly constant. Hence the inequality of which we have spoken.

In the chronometer the balance is so constructed that its figure alters when the temperature varies. The figure (4) represents a common form of the chronometer balance. The arc AB, which carries a weight at C, is formed of two concentric laminæ of different kinds of metal, the outer lamina being the most expansible. These two laminæ are securely united in their whole length, so that an increase of temperature necessarily distorts the arc AB into some form like AB'. Similarly for ab. Contrary effects are produced by a decrease of temperature. Thus, the moment of inertia I decreases as t increases; as e also does. And thus we are enabled, by

suitable adjustments, to make $\frac{e}{I}$, at least approximately, constant.

It would, I believe, be impossible, without some hypothesis, to determine the form which AB assumes under the influence of a change of temperature. The following suppositions are probably sufficiently near the truth to be applicable when the variations of t are not excessive.

Let us suppose the laminæ to be cylindrical and concentric, and bounded by four plane surfaces, two of which are perpendicular to the axis of the cylinder, while the other two, which form the boundaries at A and B, pass through the axis. These conditions being fulfilled, whatever the value of t may be, it is clear that the variation of form can depend on two elements only, namely the radius of the cylinder, and the angle which AB subtends at its centre. To determine these, we assume that the middle filament of each lamina expands as it would do if free.

In the normal state, let 2ϵ, $2\epsilon'$ be the thicknesses of the outer and inner laminæ respectively, r the radius of the boundary of the two laminæ, μ, μ' the coefficients of expansibility of the outer and inner laminæ ($\mu > \mu'$), θ the angle subtended at the centre.

The radii of the middle filaments are, therefore, $r + \epsilon$, $r - \epsilon'$; let their lengths be l and l', then

$$l = (r + \epsilon)\,\theta, \quad l' = (r - \epsilon')\,\theta.$$

For an increase of temperature t, let r and θ become r_1 and θ_1: then we shall have

$$l\,(1 + \mu t) = (r_1 + \epsilon)\,\theta_1, \quad l'\,(1 + \mu' t) = (r_1 - \epsilon')\,\theta_1;$$

ϵ and ϵ' being so small that their variations may be neglected.

Hence
$$\frac{r_1 + \epsilon}{r_1 - \epsilon'} = \frac{l}{l'}\,\frac{1 + \mu t}{1 + \mu' t},$$

and therefore
$$r_1 - \epsilon' = (\epsilon + \epsilon')\,\frac{l'\,(1 + \mu t)}{l\,(1 + \mu t) - l'\,(1 + \mu' t)},$$

which, as μ and μ' are very small, is approximately

$$r_1 - \epsilon' = (\epsilon + \epsilon')\,\frac{l'}{l - l'}\left\{1 - (\mu - \mu')\,\frac{l'}{l - l'}\,t\right\}.$$

When $t = 0$
$$r - \epsilon' = (\epsilon + \epsilon')\,\frac{l'}{l - l'}.$$

Consequently, for a first approximation,

$$\Delta r = -\,\Delta\mu\,\frac{r^2}{\tau}\,t \ldots\ldots\ldots\ldots\ldots (1),$$

where $\Delta r = r_1 - r$, $\Delta\mu = \mu - \mu'$, and $\tau = \epsilon + \epsilon_1$.

Again $\tau\theta_1 = l - l' + (\mu l - \mu' l')\,t.$

When $t = 0$ $\tau\theta = l - l'$;

and therefore $\tau\Delta\theta = (\mu l - \mu' l')\,t.$

But $\mu l - \mu' l' = r\theta\Delta\mu + (\mu\epsilon - \mu'\epsilon')\,\theta,$

and the last term is negligible. Therefore

$$\Delta\theta = \frac{r}{\tau}\,\theta\Delta\mu t \ldots\ldots\ldots\ldots\ldots (2).$$

We distinctly perceive from (1) and (2) why the effects of *distortion* are so considerable in comparison with those of simple expansion; it is because the expressions of Δr and $\Delta\theta$ have the small quantity τ in the denominator. AB becomes a larger arc of a smaller circle.

To apply these results: we suppose that when $t = O$ the centre of AB coincides with the central point O; and AB, being securely fastened at A, continues perpendicular at that point to the line OA, consequently its centre remains in that line. Let O' be its new position, then $OO' = -\,\Delta r$. If m_1 be the mass of AB, its moment of inertia about O was $m_1 r^2$; about O' it is $m_1 r^2 - 2m_1\Delta\mu\,\frac{r^3}{\tau}\,t$ nearly. Let G be the centre of gravity of AB'; then, in the triangle $OO'G$, we have $OG^2 = OO'^2 + (O'G)^2 + 2OO'.O'G\cos\frac{1}{2}\theta_1$, since $\angle GO'A = \frac{1}{2}\theta$. OO'^2 or $(\Delta r)^2$ may be neglected, then, approximately,

$$(OG)^2 = (O'G)^2 - 2\Delta r.O'G\cos\tfrac{1}{2}\theta,$$

and, as $$O'G = r_1\,\frac{\sin\frac{1}{2}\theta_1}{\frac{1}{2}\theta_1} = 2r\,\frac{\sin\frac{1}{2}\theta}{\theta}\ \text{nearly},$$

we have $$(OG)^2 - (O'G)^2 = -\,2r\,\frac{\sin\theta}{\theta}\,\Delta r.$$

Now the moment of inertia round O is equal to that round O' increased by $m\,\{(OG)^2 - (O'G)^2\}$; hence, finally,

$$\Delta I_1 = -\,2m_1\Delta\mu\,\frac{r^3}{\tau}\,t\left(1 - \frac{\sin\theta}{\theta}\right) \ldots\ldots\ldots (3),$$

I_1 being the moment of inertia of the arc AB.

(In accordance with the rest of the approximation the expansion of AO is neglected).

Again, we will suppose the weight at C to be a material particle, and that the angle AOC is equal to ϕ. Then, I_2 being the moment of inertia of this weight, whose mass we will denote by m_2, we shall have

$$\Delta I_2 = -\,2m_2\Delta\mu\,\frac{r^3}{\tau}\,t\,(1 - \cos\phi) \ldots\ldots\ldots (4),$$

Consequently, as the inertia of the bar OA does not undergo any sensible alteration, and as every thing which has been proved of OAB is true of Oab, we have, finally,

$$\Delta I = -4\Delta\mu\frac{r^3}{\tau}t\left\{m_1\left(1-\frac{\sin\theta}{\theta}\right)+m_2(1-\cos\phi)\right\} \dots (5).$$

It appears that the variation of e is exactly proportional to t: so that e becomes $e(1-\nu t)$, ν being some constant. Consequently we must have, in order that $\frac{e(1-\nu t)}{I+\Delta I}=\frac{e}{I}$,

$$\nu I = 4\Delta\mu\frac{r^3}{\tau}\left\{m_1\left(1-\frac{\sin\theta}{\theta}\right)+m_2(1-\cos\phi)\right\}\dots . (6).$$

In calculating the value of I we may take into account the moment of inertia of aOA; moreover, instead of the approximate expression m_1r^2 for the moment of inertia of AB, we may employ a more accurate one involving the quantity ε and ε'; the approximate expression is sufficiently accurate for the determination of ΔI.

The adjustment for compensation is effected by shifting the weight m_2 along AB; that is, by altering the value of ϕ until (6) is fulfilled. On the hypothesis we have made, the value of I for $t=0$ is not affected by the change of ϕ.

In determining the approximate expressions (5) and (6), we have neglected all terms in which $\Delta\mu$ occurs not divided by τ; all terms involving $\Delta\mu$ multiplied by ε or ε'; all terms into which any power of μ or μ' enters. In consequence of the last restriction t can only rise to the first power in the result. If this were absolutely correct it would follow that, if the compensation were effected for a particular value of t, it would subsist accurately for all values of t. For instance, if we give t equal values, positive and negative, the decrease of I in the one case ought to be equal to its increase in the other. But when t is considerable, it is found that there is a sensible deviation from this result; and, assuming that the expression for Δe does not in any perceptible manner involve powers of t, it follows that that of ΔI must do so. Any term involving t^2 (and, *a fortiori*, any higher powers of that quantity), must be very small, since t always occurs multiplied by μ or μ'; but it may, nevertheless, sensibly affect the chronometer's daily rate. On the usual construction, the balance oscillates 216,000 times in twenty-four hours. Consequently a very slight change in the moment of inertia of the balance will become perceptible in that period.

In order to obviate the consequent error, it has been proposed by Mr. Dent, a distinguished chronometer-maker of the present day, to alter the form of the balance. Fig. (5) represents one of those which he proposes to substitute for that in common use. It would be easy to determine the corresponding expression for ΔI, to the degree of approximation of our previous results. As, however, the comparison of the merits of the two forms must depend on the terms involving t^2, it may be well to reserve it for another opportunity. If there appears reason to believe that our hypotheses represent the facts with sufficient accuracy to encourage us to proceed farther, I hope to resume the subject in the next number of the *Journal*. R. L. E.

X.—ON A PROBLEM IN PRECESSION AND NUTATION.

In Professor Airy's *Tract on Precession and Nutation*, p. 203, there is an investigation of the angular acceleration which the disturbing force of the Sun tends to impress upon the Earth's mass about an equatoreal diameter (the axis of z) at right angles to the plane through the axis of the Earth (the axis of x) and the line joining the centre of the Earth with the Sun. The mass is divided into elementary parallelopipeds $dxdydz$, the axis of y being at right angles to those of x and z: the integrations are performed in the order of y, x, z. Since, however, the computation may be effected rather more elegantly by taking the integrations in the order of z, y, x, or by conceiving the spheroid to be made up of circular instead of elliptical slices, the method of integration given in this note may not be without interest to students in physical astronomy.

Let A (fig. 6) be the Earth's centre, AB the semi-axis of the spheroid, S the attracting body. Let P be any point of the Earth in the plane BAC; draw PM perpendicular to AB, and PN to SA.

The accelerating force on a particle of the Earth at P is equal to $\frac{S}{SP^2}$ in the direction PS, which is equivalent to

$$\frac{S}{SP^2}\cdot\frac{SN}{SP}, \text{ parallel to } AS \ldots\ldots\ldots\ldots (1),$$

and to

$$\frac{S}{SP^2}\cdot\frac{PN}{SP}, \text{ parallel to } PN \ldots\ldots\ldots\ldots (2).$$

Also the accelerating force on A, the centre of the Earth, is equal to

$$\frac{S}{SA^2}, \text{ in } AS \ldots\ldots\ldots\ldots\ldots (3).$$

Let $SA = r$; then, considering the centre of the Earth to be at rest, the disturbing force on P will be, from (1) and (3), approximately,

$$\frac{S}{(r - AN)^2} - \frac{S}{r^2} = \frac{2S.AN}{r^3}, \text{ parallel to } AS;$$

and, from (2), $$\frac{S.PN}{r^3}, \text{ parallel to } PN.$$

Hence the moment of the disturbing force on a particle m at P about a line through A, perpendicular to PAS, is

$$\frac{3S}{r^3} . m . AN . PN.$$

The same is true approximately for every molecule of the spheroid of which the projection on the plane BAC coincides with P.

Let $AC = a$, $AB = c$, $\angle BAS = \theta$, $AM = x$, $MP = y$, $2z =$ length of chord through P at right angles to plane BAC. Then $AN = x \cos\theta - y \sin\theta$, $PN = x \sin\theta + y \cos\theta$; and thus we see that the moment of the disturbing force for the whole spheroid is equal to

$$\frac{3S.k}{r^3}\int_{-c}^{+c}\int_{-y'}^{+y'} \{(x^2 - y^2)\sin\theta\cos\theta + xy(\cos^2\theta - \sin^2\theta)\}\, dx.2z\,dy,$$

where $y' =$ the radius of the circular section through MP, and $k =$ the density.

It is evident that the second term of this integral must be zero, because for every $(+y)$ there is a $(-y)$, zdy being the same for both; hence the required moment

$$= \frac{3S.k}{r^3} . 2\sin\theta\cos\theta \int_{-c}^{+c}\int_{-y'}^{+y'} (x^2 - y^2)\, z\, dx\, dy.$$

Now $$\int_{-y'}^{+y'} z\,dy = \int_{-y'}^{+y'} (y'^2 - y^2)^{\frac{1}{2}}\,dy = \tfrac{1}{2}\pi y'^2;$$

and, as may easily be ascertained,

$$\int_{-y'}^{+y'} zy^2 dy = \int_{-y'}^{+y'} y^2 (y'^2 - y^2)^{\frac{1}{2}}\,dy = \tfrac{1}{8}\pi y'^4.$$

Hence the double integral is equal to

$$\int_{-c}^{+c} (x^2 . \tfrac{1}{2}\pi y'^2 - \tfrac{1}{8}\pi y'^4)\, dx$$

$$= \tfrac{1}{8}\pi \int_{-c}^{+c} y'^2 (4x^2 - y'^2)\, dx$$

$$= \tfrac{1}{8}\pi \frac{a^2}{c^2} \int_{-c}^{+c} (c^2 - x^2) \left\{ 4x^2 - \frac{a^2}{c^2}(c^2 - x^2) \right\} dx$$

$$= \frac{\pi a^2}{8c^2} \int_{-c}^{+c} \left\{ 4c^2x^2 - 4x^4 - \frac{a^2}{c^2}(c^4 - 2c^2x^2 + x^4) \right\} dx$$

$$= \frac{\pi a^2}{8c^2} \left(\tfrac{8}{3} c^5 - \tfrac{8}{5} c^5 - 2a^2c^3 + \tfrac{4}{3} a^2c^3 - \tfrac{2}{5} a^2c^3 \right)$$

$$= \frac{\pi a^2}{8} c \,.\, \frac{16}{15} (c^2 - a^2) = \frac{2\pi}{15} a^2 c\, (c^2 - a^2).$$

Hence the required moment

$$= \frac{3S}{r^3} \cdot \frac{4\pi}{15}\, ka^2c\, (c^2 - a^2) \sin\theta \cos\theta.$$

The sign of this result shews that the effect of the Sun's attraction tends to *increase* the angle θ.

The moment of inertia of the spheroid about the axis through A, perpendicular to the area BAC, is (we know) equal to

$$\frac{4\pi}{15}\, ka^2c\, (a^2 + c^2).$$

Hence the angular acceleration is equal to

$$\frac{3S}{r^3} \cdot \frac{a^2 - c^2}{a^2 + c^2} \,.\, \sin\theta \,.\, \cos\theta.$$

W. W.

XI.—NOTES ON MAGNETISM. NO. II.

By R. L. Ellis, M.A. Fellow of Trinity College.

In order to a distinct understanding of the results obtained in the last number of the *Journal*, it will be desirable to consider the established conventions with respect to the signs of the symbols which we had occasion to employ.

North magnetism is assumed to be positive; hence, of course, south magnetism must be considered as negative.

The measure M of the magnetic power of a bar magnet is, as we have seen, equal to $\int \mu s ds$, μ being the magnetism of the element ds, which is situated at a distance from the origin equal to s. The limits of the integral are such as to include the whole length of the magnet.

The position of the origin is arbitrary: we may conveniently place it at the centre of the magnet, but the value of

$\int \mu s ds$ is the same whether this be done or any other point be taken. For let the origin be shifted through a distance a, so that $s = s' - a$, then

$$\int \mu s ds = \int \mu (s' - a) ds' = \int \mu s' ds' - a \int \mu ds' :$$

and as all the integrals extend throughout the length of the magnet $\int \mu ds' = 0$, and therefore

$$\int \mu s ds = \int \mu s' ds' \quad \text{or} \quad M = M',$$

which was to be proved.

But the value of $\int \mu s ds$ changes its sign if the direction in which s is measured changes. Let l be the length of the magnet; then, s being measured in one direction, say from left to right, we have

$$M = \int_0^l \mu s ds.$$

Now suppose that $s' = l - s$, then the limits are interchanged and $ds' = -ds$; consequently

$$M = \int_0^l \mu (l - s') ds' = -\int_0^l \mu s' ds' = -M',$$

s' being measured in the direction opposite to that of s, or from right to left.

The magnetism of a magnet may thus be always represented by a positive quantity.

Any two points in the axis of a magnet may be taken as its poles. But although the position of the poles is matter of convention, yet relatively to one another, one is the north and the other the south pole.

The physical character by which they are distinguished is this: if a particle of north magnetism be placed in the prolongation of the axis from south to north, it is repelled from the magnet. Contrariwise, if it be placed in the prolongation of the axis towards the south. Further, we must integrate $\mu s ds$ from south to north, *i.e.* s must be taken as positive when μds lies to the north of the origin, in order that M may be positive. This may be shown by supposing a particle of north magnetism m placed in the prolongation of the axis towards the north, and at a distance r from the centre of the magnet. If we assume that from south to north is positive, the action of the magnet on m is $\frac{2Mm}{r^3}$; and as this action is repulsive its expression will be positive, and therefore M is so. If we had assumed from north to south to be positive, the action of the magnet would have been represented by $-\frac{2Mm}{r^3}$,

and as this is positive, M will necessarily be negative. So that, in order to make the measure of the magnet's power positive, we must take the direction $S...N$ as positive.

Consequently the angle θ must be measured from it. We suppose it measured in the usual manner, viz. in the *unscrew* direction.

The general expression for the moment of rotation due to the action of one magnet on another is much simplified when the two magnets are supposed to lie in one plane.

The dihedral angle χ is then zero, and consequently the equation

$$L = \frac{MM'}{R^3}\{1 + 3\cos^2\theta - (2\cos\theta\cos\theta' - \sin\theta\sin\theta'\cos\chi)^2\}^{\frac{1}{2}}$$

becomes

$$L = \frac{MM'}{R^3}\{1 + 3\cos^2\theta - 4\cos^2\theta(1 - \sin^2\theta') + 4\sin\theta'\cos\theta\sin\theta\cos\theta' - \sin^2\theta\sin^2\theta'\}^{\frac{1}{2}}.$$

The quantity between the brackets is equal to

$$4\sin^2\theta'\cos^2\theta + 4\sin\theta'\cos\theta \,.\, \sin\theta\cos\theta' + \sin^2\theta\cos^2\theta'.$$

Consequently

$$L = \frac{MM'}{R^3}(\sin\theta\cos\theta' + 2\sin\theta'\cos\theta).$$

This result may be readily established by an independent process, which the reader will find no difficulty in supplying. The last result may be put in the following form:

$$L = \frac{MM'}{2R^3}\{3\sin(\theta + \theta') - \sin(\theta - \theta')\}.$$

Professor Lloyd, in the 19th volume of the *Memoirs of the Royal Irish Academy*, has investigated this case of the mutual action of two magnets. His result is, (mutatis mutandis)

$$L = \frac{MM'}{2R^3}\{\sin(\theta + \theta') - 3\sin(\theta - \theta')\}.$$

This differs from the last written result, merely because, in the Professor's analysis, θ and θ' are measured in opposite directions. If we replace θ in Prof. Lloyd's result by $2\pi - \theta$, it becomes

$$L = \frac{MM'}{2R^3}\{3\sin(\theta + \theta') - \sin(\theta - \theta')\},$$

as before.

The general formula affords a simple solution of the following problem. The position of a magnet, and that of the centre of a needle being given, to place the needle in the

position in which the moment of rotation due to the action of the magnet is a maximum.

By the formula established in the last number of the *Journal*, we have

$$L = \frac{MM'}{R^3} \{1 + 3 \cos^2 \theta - (3 \cos \theta \cos \theta' - \cos \phi)^2\}^{\frac{1}{2}}.$$

We suppose the magnet and needle to be in the same plane. In fig. (7) let O be the centre, SON the line of the axis of the magnet, C the centre of the needle. Project C on ON in D, take $OE = 2OD$, draw EN' perpendicular to SN meeting ON', which is at right angles to OC in N', CN' is the line in which the axis of the needle must be placed, its north pole being turned towards N'.

In order to prove this, we have only to remark that the angle θ or CON is constant, the position of C being given; consequently the condition to be fulfilled, in order that the moment of rotation L may be a maximum, is

$$3 \cos \theta \cos \theta' - \cos \phi = 0.$$

Now as ϕ is the angle between $N'C$ and SN, we have

$$ED = CN' \cos \phi,$$

and therefore $\qquad OD = \frac{1}{3} CN' \cos \phi.$

Again, $\qquad OC = CN' \cos \theta'$ and $OD = OC \cos \theta.$

Hence $\qquad OD = CN' \cos \theta \cos \theta'.$

Consequently $\qquad 3 \cos \theta \cos \theta' - \cos \phi = 0,$

or the required condition is fulfilled.

There are three particular cases worth noticing:

(1) $\theta = 0$. In this case C lies in the axis ON, D coincides with it, and ON' is perpendicular to ON, and therefore parallel to EN'. Consequently the point N' is removed to an infinite distance, and CN' is therefore perpendicular to ON. The corresponding value of L is $\frac{2MM'}{R^3}$, and is the maximum maximorum.

(2) $\theta = \theta'$. In this case the magnet and needle are parallel to one another. The quadrilateral $CDEN'$ is a parallelogram, EN' is equal to DC, and consequently

$$\tan COD : \tan N'OE :: EO : OD :: 2 : 1.$$

But $\qquad COD = \theta$ and $N'OE = \frac{\pi}{2} - \theta,$

since CON' is a right angle. Consequently

$$\tan \theta = 2 \cot \theta,$$

$$\text{or} \quad \tan \theta = \sqrt{2}.$$

The corresponding value of $3\cos^2\theta$ is therefore unity; and consequently we have, in this case,

$$L = \sqrt{2}\,\frac{MM'}{R^3} \quad (\theta = 54^\circ\,44'\,8'').$$

(3) $\theta = \frac{\pi}{2}$. Here D (and therefore E) coincides with O, while ON' lies in the axis OS'. Consequently N' is at O, and the needle is therefore again perpendicular to the magnet. In this case

$$L = \frac{MM'}{R^3},$$

The first and third cases were noticed in the last number of the *Journal*. In the second, the value of L is a mean proportional between what is in the other two cases, in the last of which it is a minimum maximorum.

If we were required, for a given position of C, to find the position in which the needle would be in equilibrium, or the moment L equal to zero, we might have recourse to Gauss's construction already mentioned; for if the needle be placed along the line in which the magnet tends to attract or repel C, as the dimensions of the needle are small, every element would approximately be attracted or repelled along this line, and therefore the total action would be destroyed by the resistance of C.

Thus there are always two directions for every position of C'; one of maximum moment and the other of equilibrium: these two directions are at right angles to one another.

XII.—MATHEMATICAL NOTE.

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx.$$

This definite integral is evaluated in a curious manner by M. Bertrand in *Liouville's Journal*. The demonstration I am about to give of his result, is somewhat different in form from that which he made use of.

The method employed in an ingenious paper which appeared in the last volume of the *Journal* (III. p. 188), will apply to the integral we are about to consider.

Let $$fu = \int_0^1 \frac{\log(1+ux)}{1+x^2}\,dx.$$

Then $$\frac{d}{du} fu = \int_0^1 \frac{x\,dx}{(1+ux)(1+x^2)}.$$

Now $$\frac{x}{1+x^2} + \frac{u}{1+u^2} = (1+ux)\,\frac{x+u}{(1+x^2)(1+u^2)}.$$

Consequently
$$\frac{x}{(1+ux)(1+x^2)} = \frac{x+u}{(1+x^2)(1+u^2)} - \frac{u}{(1+ux)(1+u^2)},$$
and therefore
$$\frac{d}{du}fu = \frac{1}{1+u^2}\int_0^1 \frac{xdx}{1+x^2} + \frac{u}{1+u^2}\int_0^1 \frac{dx}{1+x^2} - \frac{u}{1+u^2}\int_0^1 \frac{dx}{1+ux}$$
$$= \tfrac{1}{2}\log 2\,\frac{1}{1+u^2} + \frac{\pi}{4}\,\frac{u}{1+u^2} - \frac{\log(1+u)}{1+u^2}.$$

Integrate for u, from 0 to 1,
$$f(1) - f(0) = \frac{\pi}{8}\log 2 + \frac{\pi}{8}\log 2 - f(1).$$

But $f(0) = 0$, since $\log 1 = 0$. Therefore
$$f(1) = \frac{\pi}{8}\log 2.$$

But $$f(1) = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx.$$

Therefore $$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2.$$

The singularity of this method, and its applicability in other cases give it interest: but, as the writer of the paper already noticed pointed out to me, the integral may be got by assuming $x = \tan y$; it then becomes
$$\int_0^{\frac{\pi}{4}} \log(1+\tan y)\,dy,$$
and, by his fundamental equation,
$$\int_0^{\frac{\pi}{4}} \log(1+\tan y)\,dy = \int_0^{\frac{\pi}{4}} \log\left\{1+\tan\left(\frac{\pi}{4}-y\right)\right\}dy$$
$$1+\tan\left(\frac{\pi}{4}-y\right) = 1 + \frac{1-\tan y}{1+\tan y} = \frac{2}{1+\tan y};$$
and therefore
$$2\int_0^{\frac{\pi}{4}} \log(1+\tan y)\,dy = \frac{\pi}{4}\log 2,$$
whence the truth of M. Bertrand's result is obvious. ε.

Fig. 1.

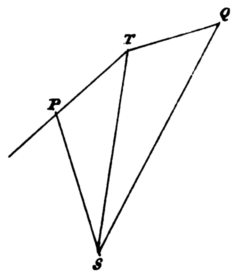


Fig. 2.

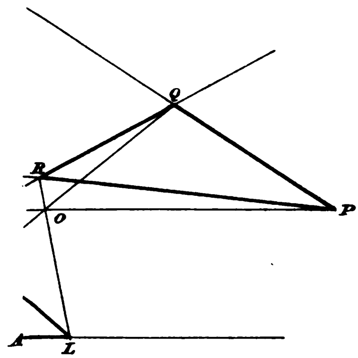
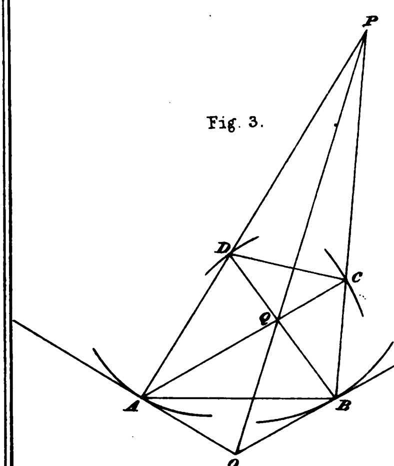


Fig. 3.



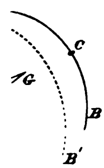
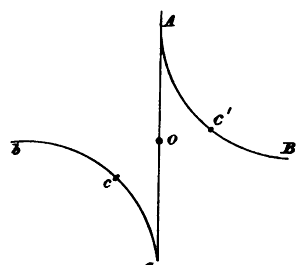


Fig. 5.



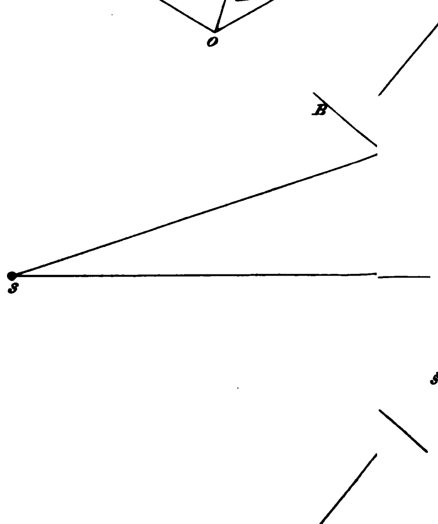
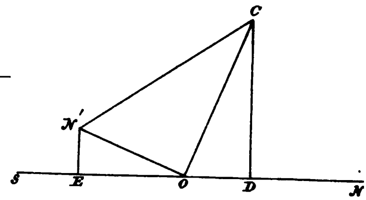


Fig. 7.



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I.—MEMOIR OF THE LATE D. F. GREGORY, M.A., FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

By R. LESLIE ELLIS, Esq., Fellow of Trinity College, Cambridge.

THE subject of the following memoir died in his thirty-first year. He had, nevertheless, accomplished enough not only to justify high expectations of his future progress in the science to which he had principally devoted himself, but also to entitle his name to a place in some permanent record.

Duncan Farquharson Gregory was born at Edinburgh in April 1813. He was the youngest son of Dr. James Gregory, the distinguished professor of Medicine, and was thus of the same family as the two celebrated mathematicians James and David Gregory. The former of these, his direct ancestor, is familiarly remembered as the inventor of the telescope which bears his name; he lived in an age of great mathematicians, and was not unworthy to be their contemporary.

Of the early years of Mr. Gregory's life but little need be said. The peculiar bent of his mind towards mathematical speculations does not appear to have been perceived during his childhood; but, in the usual course of education, he shewed much facility in the acquisition of knowledge, a remarkably active and inquiring mind, and a very retentive memory. It may, perhaps, be mentioned here, that his father, whom he lost before he was seven years old, used to predict distinction for him; and was so struck with his accurate information and clear memory, that he had pleasure in conversing with him, as with an equal, on subjects of history and geography. In his case, as in many others, ingenuity in little mechanical contrivances seems to have preceded, and indicated the developement of a taste for abstract science.

Two years of his life were passed at the Edinburgh Academy; when he left it, being considered too young for the University, he went abroad and spent a winter at a pri-

vate academy in Geneva. Here his talent for mathematics attracted attention; in geometry, as well as in classical learning, he had already made distinguished progress at Edinburgh.

The following winter he attended classes at the University of Edinburgh, and soon became a favourite pupil of Professor Wallace's, under whose tuition he made great advances in the higher parts of mathematics. The Professor formed the highest hopes of Mr. Gregory's future eminence: those who long afterwards saw them together in Cambridge, speak with much interest of the delighted pride he shewed in his pupil's success and increasing reputation.

In 1833, Mr. Gregory's name was entered at Trinity College in the University of Cambridge, and shortly afterwards he went to reside there. He brought with him a very unusual amount of knowledge on almost all scientific subjects: with Chemistry he was particularly well acquainted, so much so that he had been at Cambridge but a few months when it was proposed to him by one of the most distinguished men in the University to act as assistant to the professor of Chemistry; which for some time he did. Indeed, it is impossible to doubt that, had not other pursuits engaged his attention, he might have achieved a great reputation as a chemist. He was one of the founders of the Chemical Society in Cambridge, and occasionally gave lectures in their rooms.

He had also a very considerable knowledge of botany, and indeed of many subjects which he seemed never to have studied systematically: he possessed in a remarkable degree the power of giving a regular form, and, so to speak, a unity to knowledge acquired in fragments.

All these tastes and habits of thought Mr. Gregory cultivated, to a certain extent, during the first years of his residence in Cambridge, of course in subordination to that which was the end principally in view in his becoming a member of the University, namely, the study of mathematics and natural philosophy.

He became a Bachelor of Arts in 1837, having taken high mathematical honours: more, however, might, we may believe, have been effected in this respect, had his activity of mind permitted him to devote himself more exclusively to the prescribed course of study.

From henceforth he felt himself more at liberty to follow original speculations, and, not many months after taking his degree, turned his attention to the general theory of the combination of symbols.

It may be well to say a few words of the history of this part of mathematics.

One of the first results of the differential notation of Leibnitz, was the recognition of the analogy of differentials and powers. For instance, it was readily perceived that

$$\frac{d^{m+n}}{dx^{m+n}}y = \frac{d^m}{dx^m}\frac{d^n}{dx^n}y,$$

or, supposing the y to be *understood*, that

$$\left(\frac{d}{dx}\right)^{m+n} = \left(\frac{d}{dx}\right)^m \left(\frac{d}{dx}\right)^n,$$

just as in ordinary algebra we have, a being any quantity,

$$a^{m+n} = a^m a^n.$$

This, and one or two other remarks of the same kind, were sufficient to establish an analogy between $\frac{d}{dx}$ the symbol of differentiation and the ordinary symbols of algebra. And it was not long afterwards remarked that a corresponding analogy existed between the latter class of symbols and that which is peculiar to the calculus of finite differences. It was inferred from hence that theorems proved to be true of combinations of ordinary symbols of quantity, might be applied by analogy to the differential calculus and to that of finite differences. The meaning and interpretation of such theorems would of course be wholly changed by this kind of transfer from one part of mathematics to another, but their form would remain unchanged. By these considerations many theorems were suggested, of which it was thought almost impossible to obtain direct demonstrations. In this point of view the subject was developed by Lagrange, who left undemonstrated the results to which he was led, intimating, however, that demonstrations were required. Gradually, however, mathematicians came to perceive that the analogy with which they were dealing, involved an essential identity; and thus results, with respect to which, if the expression may be used, it had only been felt that they must be true, were now actually seen to be so. For, if the algebraical theorems by which these results were suggested, were true, *because* the symbols they involve represented quantities, and such operations as may be performed on quantities, then indeed the analogy would be altogether precarious. But if, as is really the case, these theorems are true, in virtue of certain fundamental laws of combination, which hold both for algebraical

symbols, and for those peculiar to the higher branches of mathematics, then each algebraical theorem and its analogue constitute, in fact, only one and the same theorem, except *quoad* their distinctive interpretations, and therefore a demonstration of either is in reality a demonstration of both.*

The abstract character of these considerations is doubtless the reason why so long a time elapsed before their truth was distinctly perceived. They would almost seem to require, in order that they may be readily apprehended, a peculiar faculty—a kind of mental *disinvoltura* which is by no means common.

Mr. Gregory, however, possessed it in a very remarkable degree. He at once perceived the truth and the importance of the principles of which we have been speaking, and proceeded to apply them with singular facility and fearlessness.

It had occurred to two or three distinguished writers that the analogy, as it was called, of powers, differentials, &c., might be made available in the solution of differential equations, and of equations in finite differences.

This idea, however, probably from some degree of doubt as to the legitimacy of the methods which it suggested, had not been fully or clearly developed: it seems to have been chiefly employed as affording a convenient way of expressing solutions already obtained by more familiar considerations.

To this branch of the subject Mr. Gregory directed his attention, and from the general views of the laws of combination of symbols already noticed, deduced in a regular and systematic form, methods of solution of a large and important class of differential equations (linear equations with constant coefficients, whether ordinary or partial) of systems of such equations existing simultaneously, of the corresponding classes of equations in finite and mixed differences; and lastly, of many functional equations. The steady and unwavering apprehension of the fundamental principle which pervades all these applications of it, gives them a value quite independent of that which arises from the facility of the methods of solution which they suggest.

The investigations of which I have endeavoured to illustrate the character and tendency, appeared from time to time in the *Cambridge Mathematical Journal*.

* The values of certain definite integrals are to be looked upon as merely arithmetical results; in such cases we are not at liberty to replace the constants involved in the definite integrals by symbols of operation. In other cases we are at liberty to do so, and this remarkable application of the principles stated in the text, has already led Mr. Boole of Lincoln, with whom it seems to have originated, to several curious conclusions.

In this periodical publication Mr. Gregory took much interest. He had been active in establishing it, and continued to be its editor, except for a short interval, from the time of its first appearance in the autumn of 1837, until a few months before his death. For this occupation he was for many reasons well qualified; his acquaintance with mathematical literature was very extensive, while his interest in all subjects connected with it was not only very strong, but also singularly free from the least tinge of jealous or personal feeling. That which another had done or was about to do, seemed to give him as much pleasure as if he himself had been the author of it, and this even when it related to some subject which his own researches might seem to have appropriated.

This trait, as the recollections of those who knew him best will bear me witness, was intimately connected with his whole character, which was in truth an illustration of the remark of a French writer, that to be free from envy is the surest indication of a fine nature.

To the *Cambridge Mathematical Journal*, Mr. Gregory contributed many papers beside those which relate to the researches already noticed. In some of these he developed certain particular applications of the principles he had laid down in an Essay on the Foundations of Algebra, presented to the Royal Society of Edinburgh in 1838, and printed in the fourteenth volume of their Transactions. I may particularly mention a paper on the curious question of the logarithms of negative quantities, a question which, it is well known, has often been discussed among mathematicians, and which even now does not appear to be entirely settled.

In 1840, Mr. Gregory was elected Fellow of Trinity College; in the following year he became Master of Arts, and was appointed to the office of moderator, that is, of principal mathematical examiner. His discharge of the duties of this office (which is looked upon as one of the most honourable of those which are accessible to the younger members of the University) was distinguished by great good sense and discretion.

In the close of the year 1841, Mr. Gregory produced his "Collection of Examples of the Processes of the Differential and Integral Calculus;" a work which required, and which manifests much research, and an extensive acquaintance with mathematical writings. He had at first only wished to superintend the publication of a second edition of the work with a similar title, which appeared more than twenty-five years

since, and of which Messrs. Herschel, Peacock, and Babbage were the authors. Difficulties, however, arose, which prevented the fulfilment of this wish, and it is not perhaps to be regretted that Mr. Gregory was thus led to undertake a more original design. It is well known that the earlier work exercised a great and beneficial influence on the studies of the University, nor was it in any way unworthy of the reputation of its authors. The original matter contributed by Sir John Herschel is especially valuable. Nevertheless, the progress which mathematical science has since made, rendered it desirable that another work of the same kind should be produced, in which the more recent improvements of the calculus might be embodied.

Since the beginning of the century, the general aspect of mathematics has greatly changed. A different class of problems from that which chiefly engaged the attention of the great writers of the last age has arisen, and the new requirements of natural philosophy have greatly influenced the progress of pure analysis. The mathematical theories of heat, light, electricity, and magnetism, may be fairly regarded as the achievement of the last fifty years. And in this class of researches an idea is prominent, which comparatively occurs but seldom in purely dynamical enquiries. This is the idea of discontinuity. Thus, for instance, in the theory of heat, the conditions relating to the surface of the body whose variations of temperature we are considering, form an essential and peculiar element of the problem; their peculiarity arises from the discontinuity of the transition from the temperature of the body to that of the space in which it is placed. Similarly, in the undulatory theory of light, there is much difficulty in determining the conditions which belong to the bounding surfaces of any portion of ether; and although this difficulty has, in the ordinary applications of the theory, been avoided by the introduction of proximate principles, it cannot be said to have been got rid of.

The power, therefore, of symbolizing discontinuity, if such an expression may be permitted, is essential to the progress of the more recent applications of mathematics to natural philosophy, and it is well known that this power is intimately connected with the theory of definite integrals. Hence the principal importance of this theory, which was altogether passed over in the earlier collection of examples.

Mr. Gregory devoted to it a chapter of his work, and noticed particularly some of the more remarkable applications of definite integrals to the expression of the solutions of

partial differential equations. It is not improbable that in another edition he would have developed this subject at somewhat greater length. He had long been an admirer of Fourier's great work on heat, to which this part of mathematics owes so much; and once, while turning over its pages, remarked to the writer,—"All these things seem to me to be a kind of mathematical paradise."

In 1841, the mathematical Professorship at Toronto was offered to Mr. Gregory: this, however, circumstances induced him to decline. Some years previously he had been a candidate for the Mathematical Chair at Edinburgh.

His year of office as moderator ended in October 1842. In the University Examination for Mathematical Honours in the following January, he, however, in accordance with the usual routine, took a share, with the title of examiner,—a position little less important, and very nearly as laborious, as that of moderator. Besides these engagements in the University, he had been for two or three years actively employed in lecturing and examining in the College of which he was a Fellow. In the fulfilment of these duties, he shewed an earnest and constant desire for the improvement of his pupils, and his own love of science tended to diffuse a taste for it among the better order of students. He had for some time meditated a work on Finite Differences, and had commenced a treatise on Solid Geometry, which, unhappily, he did not live to complete. In the midst of these various occupations, he felt the earliest approaches of the malady which terminated his life.

The first attack of illness occurred towards the close of 1842. It was succeeded by others, and in the spring of 1843, he left Cambridge never to return again. He had just before taken part in a college examination, and notwithstanding severe suffering, had gone through the irksome labour of examining with patient energy and undiminished interest.

Many months followed of almost constant pain. Whenever an interval of tolerable ease occurred, he continued to interest himself in the pursuits to which he had been so long devoted; he went on with the work on Geometry, and, but a little while before his death, commenced a paper on the analogy of differential equations and those in finite differences. This analogy it is known that he had developed to a great length; unfortunately, only a portion of his views on the subject can now be ascertained.

At length, on the 23d February 1844, after sufferings, on

which, notwithstanding the admirable patience with which they were borne, it would be painful to dwell, his illness terminated in death. He had been for a short time aware that the end was at hand, and, with an unclouded mind, he prepared himself calmly and humbly for the great change; receiving and giving comfort and support from the thankful hope that the close of his suffering life here, was to be the beginning of an endless existence of rest and happiness in another world. He retained to the last, when he knew that his own connection with earthly things was soon to cease, the unselfish interest which he had ever felt in the pursuits and happiness of those he loved.

A few words may be allowed about a character where rare and sterling qualities were combined. His upright, sincere, and honourable nature secured to him general respect. By his intimate friends, he was admired for the extent and variety of his information, always communicated readily, but without a thought of display,—for his refinement and delicacy of taste and feeling,—for his conversational powers and playful wit; and he was beloved by them for his generous, amiable disposition, his active and disinterested kindness, and steady affection. And in this manner his high-toned character acquired a moral influence over his contemporaries and juniors, in a degree remarkable in one so early removed.

To this brief history, little more is to be added; for though it is impossible not to indulge in speculations as to all that Mr. Gregory might have done in the cause of science and for his own reputation, had his life been prolonged, yet such speculations are necessarily too vague to find a place here; and even were it not so, it would perhaps be unwise to enter on a subject so full of sources of unavailing regret.

II.—ON THE PARTIAL DIFFERENTIAL EQUATIONS TO A FAMILY OF ENVELOPS.

By W. Walton, M.A. Trinity College.

The subject of this paper is the following general problem: "To investigate the partial differential equation of the envelop of a surface, the equation to which involves three variable parameters, the parameters themselves being subject to two unknown equations of relation."

The solution of problems of this class has not yet, as far as I am aware, been attempted by symmetrical methods: my object is to supply this deficiency. I shall begin with the

particular cases of Developable and Tubular surfaces: I shall then proceed to the consideration of the problem in all its generality. The arguments which I have laid down in the discussion of the two individual problems, and in the general one, are so precisely similar as to appear tautological: I have chosen, however, so to express myself, in order that each of the divisions of the paper may be separately intelligible.

In the following researches I shall put, for the sake of brevity,

$$\frac{du}{dx} = a, \quad \frac{du}{dy} = b, \quad \frac{du}{dz} = c,$$

$$\frac{d^2u}{dx^2} = a', \quad \frac{d^2u}{dy^2} = b', \quad \frac{d^2u}{dz^2} = c',$$

$$\frac{d^2u}{dydz} = a'', \quad \frac{d^2u}{dzdx} = b'', \quad \frac{d^2u}{dxdy} = c''.$$

1. Let $x, y, z,$ be the co-ordinates of any point of a developable surface; $\alpha, \beta, \gamma,$ the variable parameters. Then

$$\alpha x + \beta y + \gamma z = 1 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$x d\alpha + y d\beta + z d\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

$\alpha, \beta, \gamma,$ being subject to two equations

$$F(\alpha, \beta, \gamma) = 0, \quad f(\alpha, \beta, \gamma) = 0 \ldots\ldots\ldots (3).$$

From (1) and (2) there is

$$\alpha dx + \beta dy + \gamma dz = 0 \ldots\ldots\ldots\ldots (4).$$

Suppose $u = 0$ to be the equation to the developable surface: then we shall have also

$$adx + bdy + cdz = 0 \ldots\ldots\ldots\ldots (5).$$

By the aid of an indeterminate multiplier λ we shall get from (4) and (5), observing that by virtue of (1), (2), (3), x and y may be regarded as independent variables,

$$\alpha = \frac{a}{\lambda}, \quad \beta = \frac{b}{\lambda}, \quad \gamma = \frac{c}{\lambda} \ldots\ldots\ldots (6).$$

Now the only equations connecting $\alpha, \beta, \gamma, x, y, z,$ with $d\alpha, d\beta, d\gamma,$ are (2) and the differentials of (3); all which three equations are satisfied identically by putting

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0,$$

without subjecting to any limitation the absolute or relative values of $x, y, z, \alpha, \beta, \gamma$. Differentiating, then, equations (6) on this hypothesis, we get

$$\frac{d\lambda}{\lambda} = \frac{da}{a} = \frac{1}{a} \cdot (a'dx + c''dy + b''dz),$$

$$\frac{d\lambda}{\lambda} = \frac{db}{b} = \frac{1}{b} \cdot (b'dy + a''dz + c''dx),$$

$$\frac{d\lambda}{\lambda} = \frac{dc}{c} = \frac{1}{c} \cdot (c'dz + b''dx + a''dy).$$

Eliminating dy and dz by cross-multiplication, we get

$$abc \cdot \frac{d\lambda}{\lambda} \cdot \{a(b'c' - a''^2) + b(a''b'' - c'c'') + c(c''a'' - b'b'')\} = Vdx \ldots (7),$$

where $\quad V = a'b'c' - a'a''^2 - b'b''^2 - c'c''^2 + 2a''b''c''.$

Observing that V is a symmetrical function of a', b', c', a'', b'', c'', it is evident that we shall have also

$$abc \cdot \frac{d\lambda}{\lambda} \cdot \{b(c'a' - b''^2) + c(b''c'' - a'a'') + a(a''b'' - c'c'')\} = Vdy \ldots (8),$$

$$abc \cdot \frac{d\lambda}{\lambda} \cdot \{c(a'b' - c''^2) + a(c''a'' - b'b'') + b(b''c'' - a'a'')\} = Vdz \ldots (9).$$

Multiplying these equations (7), (8), (9), by a, b, c, respectively, adding, and attending to (5), we get

$$a^2(b'c' - a''^2) + b^2(c'a' - b''^2) + c^2(a'b' - c''^2)$$
$$+ 2bc(b''c'' - a'a'') + 2ca(c''a'' - b'b'') + 2ab(a''b'' - c'c'') = 0,$$

as the symmetrical form of the partial differential equation of developable surfaces.

2. A Tubular surface is the envelop of a series of spheres of invariable radius, the centres of which lie in a curve of which the equations are given. Let ρ be the radius of each sphere; α, β, γ, the co-ordinates of the centre of any one of the spheres: then, x, y, z, being the co-ordinates of any point of the envelop, we shall have

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = \rho^2 \ldots . (1),$$

$$(x - \alpha)\, d\alpha + (y - \beta)\, d\beta + (z - \gamma)\, d\gamma = 0 \ldots . (2).$$

The quantities α, β, γ, are subject to two equations

$$F(\alpha, \beta, \gamma) = 0, \quad f(\alpha, \beta, \gamma) = 0 \ldots\ldots (3).$$

From (1) and (2) we get

$$(x - \alpha)\, dx + (y - \beta)\, dy + (z - \gamma)\, dz = 0 \ldots . (4).$$

Suppose $u = 0$ to be the equation to the tubular surface; then we shall have also

$$adx + bdy + cdz = 0 \ldots\ldots\ldots\ldots (5).$$

Now α, β, γ, x, y, z, being connected by the equations (1), (2), (3), it is evident that α, β, γ, z, may be regarded as functions of two independent variables x and y: we have then, from (4) and (5), by the aid of an indeterminate multiplier λ,

$$\lambda a + x - \alpha = 0, \quad \lambda b + y - \beta = 0, \quad \lambda c + z - \gamma = 0 \ldots (6).$$

Now the only equations connecting α, β, γ, x, y, z, with $d\alpha$, $d\beta$, $d\gamma$, are (2) and the differentials of (3); but all these three equations are satisfied identically by putting

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0,$$

without subjecting to any limitation the absolute or relative values of x, y, z, α, β, γ: differentiating then equations (6) on this hypothesis, we get

$$-d\lambda = \frac{\lambda da + dx}{a} = \frac{\lambda db + dy}{b} = \frac{\lambda dc + dz}{c},$$

and therefore, performing the differentiations,

$$(1 + \lambda a')\, dx + \lambda c'' dy + \lambda b'' dz = -a d\lambda,$$
$$(1 + \lambda b')\, dy + \lambda a'' dz + \lambda c'' dx = -b d\lambda,$$
$$(1 + \lambda c')\, dz + \lambda b'' dx + \lambda a'' dy = -c d\lambda.$$

Eliminating dy and dz from these three equations, we get

$$-\frac{Vdx}{d\lambda} = a + \lambda\{a(b' + c') - bc'' - cb''\} + \lambda^2\{a(b'c' - a''^2) + a''(bb'' + cc'') - bc'c'' - cb'b''\} \ldots (7),$$

where $V = (1 + \lambda a')(1 + \lambda b')(1 + \lambda c') - \lambda^2(a''^2 + b''^2 + c''^2) - \lambda^3(a'a''^2 + b'b''^2 + c'c''^2 - 2a''b''c'')$,

a symmetrical function of a', b', c', a'', b'', c''. We must have, therefore, also

$$-\frac{Vdy}{d\lambda} = b + \lambda\{b(c' + a') - ca'' - ac''\} + \lambda^2\{b(c'a' - b''^2) + b''(cc'' + aa'') - ca'a'' - ac'c''\} \ldots (8),$$

$$-\frac{Vdz}{d\lambda} = c + \lambda\{c(a' + b') - ab'' - ba''\} + \lambda^2\{c(a'b' - c''^2) + c''(aa'' + bb'') - ab'b'' - ba'a''\} \ldots (9).$$

Multiplying equations (7), (8), (9), by a, b, c, respectively, adding, and paying attention to (5), we get

$$0 = a^2 + b^2 + c^2 + \lambda\{a'(b^2 + c^2) + b'(c^2 + a^2) + c'(a^2 + b^2) - 2a''bc - 2b''ca - 2c''ab\}$$
$$+ \lambda^2\{a^2(b'c' - a''^2) + b^2(c'a' - b''^2) + c^2(a'b' - c''^2) + 2bc(b''c'' - a'a'') + 2ca(c''a'' - b'b'') + 2ab(a''b'' - c'c'')\} = 0:$$

but, from (1) and (6),

$$\lambda^2 = \frac{\rho^2}{a^2 + b^2 + c^2};$$

hence we obtain for the symmetrical form of the differential equation to tubular surfaces,

$$(a^2+b^2+c^2)^2 \pm \rho\,(a^2+b^2+c^2)^{\frac{1}{2}}\{a'(b^2+c^2)+b'(c^2+a^2)+c'(a^2+b^2) - 2a''bc - 2b''ca - 2c''ab\}$$
$$+ \rho^2\{a^2(b'c'-a''^2)+b^2(c'a'-b''^2)+c^2(a'b'-c''^2) + 2bc\,(b''c''-a'a'')+2ca\,(c''a''-b'b'')+2ab\,(a''b''-c'c'')\} = 0.$$

3. Let the equation to any surface be

$$f = f(x, y, z, \alpha, \beta, \gamma) = 0 \ldots\ldots\ldots\ldots(1),$$

and let it be proposed to find the partial differential equation to its envelop, when the parameters α, β, γ, vary under the conditions expressed by two equations

$$\phi(\alpha, \beta, \gamma) = 0, \quad \psi(\alpha, \beta, \gamma) = 0 \ldots\ldots (2),$$

the forms of ϕ and ψ not being assigned.

We shall have, also,

$$\frac{df}{d\alpha}\,d\alpha + \frac{df}{d\beta}\,d\beta + \frac{df}{d\gamma}\,d\gamma = 0 \ldots\ldots\ldots (3),$$

and therefore, from (1),

$$\frac{df}{dx}\,dx + \frac{df}{dy}\,dy + \frac{df}{dz}\,dz = 0 \ldots\ldots\ldots\ldots(4).$$

Suppose $u = 0$ to be the equation to the envelop; then

$$adx + bdy + cdz = 0 \ldots\ldots\ldots\ldots(5).$$

In what follows we shall put

$$\frac{df}{dx} = l, \quad \frac{df}{dy} = m, \quad \frac{df}{dz} = n,$$

$$\frac{d^2f}{dx^2} = l', \quad \frac{d^2f}{dy^2} = m', \quad \frac{d^2f}{dz^2} = n',$$

$$\frac{d^2f}{dydz} = l'', \quad \frac{d^2f}{dzdx} = m'', \quad \frac{d^2f}{dxdy} = n''.$$

Now $\alpha, \beta, \gamma, x, y, z$, being connected by the equations (1), (2), (3), it is evident that α, β, γ, z, may be regarded as functions of two independent variables x and y: we have then, from (4) and (5), by the aid of an indeterminate multiplier λ,

$$\lambda a + l = 0, \quad \lambda b + m = 0, \quad \lambda c + n = 0\ldots\ldots(6).$$

Now the only equations connecting $a, \beta, \gamma, x, y, z$, with da, $d\beta$, $d\gamma$, are (3) and the differentials of (2); but all these three equations are satisfied identically by putting

$$da = 0, \quad d\beta = 0, \quad d\gamma = 0,$$

without subjecting to any limitation the absolute or relative values of $x, y, z, a, \beta, \gamma$: differentiating, then, equations (6) on this hypothesis, we get

$$(\lambda a' + l')\,dx + (\lambda c'' + n'')\,dy + (\lambda b'' + m'')\,dz = -\,a d\lambda,$$
$$(\lambda b' + m')\,dy + (\lambda a'' + l'')\,dz + (\lambda c'' + n'')\,dx = -\,b d\lambda,$$
$$(\lambda c' + n')\,dz + (\lambda b'' + m'')\,dx + (\lambda a'' + l'')\,dy = -\,c d\lambda.$$

Eliminating dy and dz from these equations by cross-multiplication, we get

$$\begin{aligned} -\frac{Vdx}{d\lambda} &= a\{(\lambda b' + m')(\lambda c' + n') - (\lambda a'' + l'')^2\} \\ &+ b\{(\lambda a'' + l'')(\lambda b'' + m'') - (\lambda c' + n')(\lambda c'' + n'')\} \\ &+ c\{(\lambda c'' + n'')(\lambda a'' + l'') - (\lambda b' + m')(\lambda b'' + m'')\} \ldots (7), \end{aligned}$$

where $V = (\lambda a' + l')(\lambda b' + m')(\lambda c' + n') + 2(\lambda a'' + l'')(\lambda b'' + m'')(\lambda c'' + n'')$
$- (\lambda a' + l')(\lambda a'' + l'')^2 - (\lambda b' + m')(\lambda b'' + m'')^2 - (\lambda c' + n')(\lambda c'' + n'')^2$,

which is a symmetrical function of

$$\lambda a' + l', \quad \lambda b' + m', \quad \lambda c' + n',$$
$$\lambda a'' + l'', \quad \lambda b'' + m'', \quad \lambda c'' + n'':$$

we must, therefore, have also

$$\begin{aligned} -\frac{Vdy}{d\lambda} &= b\{(\lambda c' + n')(\lambda a' + l') - (\lambda b'' + m'')^2\} \\ &+ c\{(\lambda b'' + m'')(\lambda c'' + n'') - (\lambda a' + l')(\lambda a'' + l'')\} \\ &+ a\{(\lambda a'' + l'')(\lambda b'' + m'') - (\lambda c' + n')(\lambda c'' + n'')\} \ldots (8), \end{aligned}$$

$$\begin{aligned} -\frac{Vdz}{d\lambda} &= c\{(\lambda a' + l')(\lambda b' + m') - (\lambda c'' + n'')^2\} \\ &+ a\{(\lambda c'' + n'')(\lambda a'' + l'') - (\lambda b' + m')(\lambda b'' + m'')\} \\ &+ b\{(\lambda b'' + m'')(\lambda c'' + n'') - (\lambda a' + l')(\lambda a'' + l'')\} \ldots (9). \end{aligned}$$

Multiplying (7), (8), (9), by a, b, c, respectively, adding, and regarding (5), we obtain

$$\begin{aligned} 0 &= a^2\{(\lambda b' + m')(\lambda c' + n') - (\lambda a'' + l'')^2\} \\ &+ b^2\{(\lambda c' + n')(\lambda a' + l') - (\lambda b'' + m'')^2\} \\ &+ c^2\{(\lambda a' + l')(\lambda b' + m') - (\lambda c'' + n'')^2\} \\ &+ 2bc\{(\lambda b'' + m'')(\lambda c'' + n'') - (\lambda a' + l')(\lambda a'' + l'')\} \\ &+ 2ca\{(\lambda c'' + n'')(\lambda a'' + l'') - (\lambda b' + m')(\lambda b'' + m'')\} \\ &+ 2ab\{(\lambda a'' + l'')(\lambda b'' + m'') - (\lambda c' + n')(\lambda c'' + n'')\} \ldots (10) \end{aligned}$$

Now, from equations (1) and (6), we may determine α, β, γ, λ, in terms of x, y, z, a, b, c: substituting the resulting expressions for these four quantities in the equation (10), we shall thus obtain the partial differential equation to the envelop of surface (1).

4. The transformation of the partial differential equations from the symmetrical to the unsymmetrical form is readily effected. Suppose, in fact, the equation $u = 0$ to be reduced to the form

$$u = z - f(x, y) = 0:$$

then it is easily seen that, p, q, r, s, t, denoting the partial differential coefficients of z with respect to x and y according to the usual notation,

$$\begin{array}{lll} a = -p, & b = -q, & c = 1, \\ a' = -r, & b' = -t, & c' = 0, \\ a'' = 0, & b'' = 0, & c'' = -s. \end{array}$$

If we substitute these values of the partial differential coefficients of u in the partial differential equations to the surface, we shall at once effect the proposed transformation. Thus the equation to developable surfaces becomes

$$rt - s^2 = 0;$$

and the equation to tubular surfaces assumes the form

$$(1+p^2+q^2)^2 \pm \rho(1+p^2+q^2)^{\frac{1}{2}}\{r(1+q^2)-2pqs+t(1+p^2)\}+\rho^2(rt-s^2)=0,$$

which may be seen in Moigno's *Leçons de Calcul Differential et de Calcul Integral*, tom. I. p. 478.

III.—ON THE AXIS OF SPONTANEOUS ROTATION.

When a rigid system is suddenly put in motion by the action of impulsive forces, there will under certain circumstances be a straight line, about which the system will begin to revolve as an instantaneous axis; this line is called the axis of Spontaneous Rotation. I have nowhere seen any investigation of the condition to be satisfied, in order that such an axis may exist; and this is what I now propose to supply.

Let the rigid system be subject to impulsive forces, which are reducible to three impulsive pressures, X, Y, Z, at the origin, and three impulsive couples whose moments are L, M, N.

Let V_x, V_y, V_z be the absolute velocities of a particle δm, (whose co-ordinates, measured from the centre of gravity, are x, y, z,) parallel to the axes of co-ordinates; V_x', V_y', V_z' the

velocities of the same point relative to the centre of gravity; $\overline{V}_x, \overline{V}_y, \overline{V}_z$ the velocities of the centre of gravity; $\omega_1, \omega_2, \omega_3$ the impulsive angular velocities of the system about the three axes.

Then we have these relations,

$$V_x = \overline{V}_x + V'_x, \quad V_y = \overline{V}_y + V'_y, \quad V_z = \overline{V}_z + V'_z;$$

and

$$\left.\begin{aligned} V'_x &= z\omega_2 - y\omega_3 \\ V'_y &= x\omega_3 - z\omega_1 \\ V'_z &= y\omega_1 - x\omega_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots (1).$$

Now we have for the motion about the centre of gravity the equations

$$\left.\begin{aligned} \Sigma\delta m(yV'_z - zV'_y) &= L \\ \Sigma\delta m(zV'_x - xV'_z) &= M \\ \Sigma\delta m(xV'_y - yV'_x) &= N \end{aligned}\right\} \ldots\ldots\ldots\ldots (2);$$

which, by substituting for V'_x, V'_y, V'_z the values given above, become

$$\left.\begin{aligned} \omega_1\,\Sigma\delta m(y^2+z^2) - \omega_2\,\Sigma\delta mxy - \omega_3\,\Sigma\delta mxz &= L \\ \omega_2\,\Sigma\delta m(x^2+z^2) - \omega_3\,\Sigma\delta myz - \omega_1\,\Sigma\delta mxy &= M \\ \omega_3\,\Sigma\delta m(x^2+y^2) - \omega_1\,\Sigma\delta mxz - \omega_2\,\Sigma\delta myz &= N \end{aligned}\right\} \ldots\ldots (3).$$

To simplify these equations, suppose the principal axes through the centre of gravity to be axes of co-ordinates, which may always be done without loss of generality, and call A, B, C the principal moments of inertia of the system; then we have

$$\left.\begin{aligned} \omega_1 &= \frac{L}{A} \\ \omega_2 &= \frac{M}{B} \\ \omega_3 &= \frac{N}{C} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (4).$$

Again, we have for the motion of the centre of gravity, (if m be the whole mass of the system,)

$$\left.\begin{aligned} \overline{V}_x &= \frac{X}{m} \\ \overline{V}_y &= \frac{Y}{m} \\ \overline{V}_z &= \frac{Z}{m} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (5);$$

and therefore, for the absolute velocities of any particle δm, we shall have

$$\left.\begin{aligned} V_x &= \overline{V}_x + V_x' = \frac{X}{m} + z\frac{M}{B} - y\frac{N}{C} \\ V_y &= \overline{V}_y + V_y' = \frac{Y}{m} + x\frac{N}{C} - z\frac{L}{A} \\ V_z &= \overline{V}_z + V_z' = \frac{Z}{m} + y\frac{L}{A} - x\frac{M}{B} \end{aligned}\right\} \dots\dots\dots\dots (6).$$

To determine the points which are at rest, we must put $V_x = 0$, $V_y = 0$, $V_z = 0$, and we have

$$\left.\begin{aligned} y\frac{N}{C} - z\frac{M}{B} &= \frac{X}{m} \\ z\frac{L}{A} - x\frac{N}{C} &= \frac{Y}{m} \\ x\frac{M}{B} - y\frac{L}{A} &= \frac{Z}{m} \end{aligned}\right\} \dots\dots\dots\dots (7);$$

these three equations are not independent, for if we multiply them by $\frac{L}{A}$, $\frac{M}{B}$ and $\frac{N}{C}$ respectively and add, there results

$$\frac{L.X}{A} + \frac{M.Y}{B} + \frac{N.Z}{C} = 0 \dots\dots\dots (8),$$

which is a relation between the forces and the constitution of the system, in order that there may be an axis of spontaneous rotation; and if this condition be satisfied, any two of the equations (7) will be the equations to the axis.

If the moments of inertia A, B, C, be all equal, the equation (8) becomes the condition of the forces acting on the system, having a single resultant.

It is easy to shew that the axis of spontaneous rotation is perpendicular to the direction of the resultant of the forces; for the direction-cosines of this resultant are

$$\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R};$$

and, if θ be the angle between it and the spontaneous axis, we have

$$\cos\theta = \frac{\dfrac{LX}{A} + \dfrac{MY}{B} + \dfrac{NZ}{C}}{R\sqrt{\left(\dfrac{L^2}{A^2} + \dfrac{M^2}{B^2} + \dfrac{N^2}{C^2}\right)}} = 0, \quad \text{by } (8);$$

and therefore θ is a right angle.

This proposition may be more elegantly proved thus: multiplying equations (7) by x, y, z, respectively, and adding, we have

$$xX + yY + zZ = 0,$$

which is the equation to a plane in which the spontaneous axis lies, and is perpendicular to the line whose direction-cosines are proportional to X, Y, and Z.

It is not difficult to see the physical interpretation of the condition (8). If we consider the motion of the system as made up of the motion of translation of the centre of gravity, and the rotation about the centre of gravity, then X, Y, Z, will be proportional to the direction-cosines of the path of the centre of gravity, and $\frac{L}{A}, \frac{M}{B}, \frac{N}{C}$, being respectively equal to ω_1, ω_2, ω_3, will be proportional to the direction-cosines of the instantaneous axis, about which the system revolves when it receives the impulse: consequently the equation (8) expresses that the motion of the centre of gravity is in the plane perpendicular to the axis of rotation, and it is quite clear, that unless this be case, it is impossible that the motion of any point due to the motion of the centre of gravity should be counteracted by the motion due to rotation about the centre of gravity. The equation (8) is, in fact, exactly similar to that which expresses the condition, that a system of forces shall admit of a single resultant: for the equation expressing this latter condition, implies that the resultant force acts in the plane of the resultant couple; or, which is the same thing, that the plane in which there is a tendency to translation is perpendicular to the axis about which there is a tendency to rotation.

In the particular case of a blow at a definite point, whose co-ordinates are a, b, c, we have

$$L = bZ - cY : M = cX - aZ : N = aY - bX,$$

and the condition (8) becomes

$$\frac{a}{X}\left(\frac{1}{B} - \frac{1}{C}\right) + \frac{b}{Y}\left(\frac{1}{C} - \frac{1}{A}\right) + \frac{c}{Z}\left(\frac{1}{A} - \frac{1}{B}\right) = 0 \ldots\ldots (9).$$

Let us suppose this blow to be parallel to one of the principal axes, as the axis of z for instance, and in the plane of xz; then $b = 0$, $c = 0$, $X = 0$, $Y = 0$, $L = 0$, $M = -aZ$, $N = 0$; and equations (7) become

$$z = 0,$$

$$x = -\frac{B}{ma} = -\frac{k^2}{a} \ldots\ldots\ldots\ldots (10),$$

if k be the radius of gyration with respect to the axis of x.

In this case, if the line of action of the blow were made the *axis of suspension*, the point in which the spontaneous axis meets the plane of xz, would be the *centre of oscillation* of the system. The axis thus determined is that found as the axis of Spontaneous Rotation in *Pratt's Mechanical Philosophy:* it may be seen from the preceding investigation, how very limited the application of the formula (10) must be.

It will be worth while to consider fully the case of a rigid system acted upon by a single blow: for this purpose, I shall assume the blow to act parallel to the axis of z, and in the plane of xy; but, to obtain the necessary generality, I shall no longer assume the principal axes to be axes of co-ordinates. Resuming then equations (3), and putting $\Sigma\ \delta m\ yz = D$, $\Sigma\ \delta m\ xz = E$, and $\Sigma\ \delta m\ xy = F$, we have, for the determination $\omega_1, \omega_2, \omega_3$,

$$\left.\begin{aligned} A\omega_1 - F\omega_2 - E\omega_3 &= 0 \\ B\omega_2 - D\omega_3 - F\omega_1 &= -aZ \\ C\omega_3 - E\omega_1 - D\omega_2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots (11);$$

and for the equations to the spontaneous axis,

$$\left.\begin{aligned} y\omega_3 - z\omega_2 &= 0 \\ z\omega_1 - x\omega_3 &= 0 \\ x\omega_2 - y\omega_1 &= \frac{Z}{m} \end{aligned}\right\} \ldots\ldots\ldots\ldots (12).$$

Multiplying these last equations by $\omega_1, \omega_2, \omega_3$ respectively, and adding, we see that ω_3 must $= 0$, which reduces equations (11) to the following,

$$\left.\begin{aligned} A\omega_1 - F\omega_2 &= 0 \\ B\omega_2 - F\omega_1 &= -aZ \\ E\omega_1 + D\omega_2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots (13);$$

whence

$$\left.\begin{aligned} \omega_1 &= \frac{F}{F^2 - AB} \cdot aZ \\ \omega_2 &= \frac{A}{F^2 - AB} \cdot aZ \end{aligned}\right\} \ldots\ldots\ldots\ldots (14);$$

with the condition $\quad AD + EF = 0 \ldots\ldots\ldots\ldots (15),$

and equations (12) become

$$\left.\begin{aligned} z &= 0, \\ Ax - Fy &= \frac{F^2 - AB}{ma} \end{aligned}\right\} \ldots\ldots\ldots (16).$$

On the whole, therefore, we have the condition (15), in order

that there may be a spontaneous axis; and if there be, its equations are (16).

The axis of spontaneous rotation possesses a remarkable property, the discovery of which is I believe due to Euler, and of which Lagrange has given a demonstration in the *Mécanique Analytique*; viz. that the *vis-viva* of the system with respect to this axis is a *maximum* or a *minimum*. Lagrange's proof leaves it doubtful which it is, but I believe it will appear that it is always the former. I shall proceed to obtain a general expression for the *vis-viva* of the system, from which also the truth of the above proposition may be made to appear.

By the general equation of *vis-viva*, we have

$$\begin{aligned}
\textit{vis-viva} &= \Sigma\,(XV_x + YV_y + ZV_z) \\
&= \Sigma\,\{X(\overline{V}_x + z\omega_2 - y\omega_3) + Y(\overline{V}_y + x\omega_3 - z\omega_1) \\
&\qquad\qquad + Z(\overline{V}_z + y\omega_1 - x\omega_2)\} \\
&= \Sigma\,(X\overline{V}_x + Y\overline{V}_y + Z\overline{V}_z) + \Sigma\,(L\omega_1 + M\omega_2 + N\omega_3) \\
&= \frac{X^2 + Y^2 + Z^2}{m} + \frac{L^2}{A} + \frac{M^2}{B} + \frac{N^2}{C} \;\ldots\ldots\; (17).
\end{aligned}$$

(The Σ is dropped, because X, Y, Z, L, M, N, are supposed to be the resultants of all the forces and couples acting on the system.)

From this equation, which involves no elements of space but such as depend on the system itself, we conclude that the whole *vis-viva* generated by the impulse is an absolute constant. If now we were to consider any axis in the system fixed, there would be an impulsive pressure on this axis which would *destroy vis-viva;* but, by the very nature of the axis of Spontaneous Rotation, there is no impulsive pressure upon it; consequently, if it be fixed, no *vis-viva* is destroyed, and therefore the *vis-viva* calculated with regard to this axis will be greater than for any other. H. G.

IV.—ON BRIANCHON'S HEXAGON.

By W. WALTON, M.A. Trin. Coll.

If any hexagon be circumscribed about a conic section, the three diagonals joining opposite angles will all pass through one point.

This elegant property of conic sections was first given by M. Brianchon, in the 13th *Cahier* of the *Journal de l'Ecole Polytechnique*, p. 301, where it is deduced as a corollary

from Pascal's *Hexagramme Mystique.* An analytical demonstration of this theorem in the particular case of the parabola, has been supplied by Mr. Lubbock, in the *Philosophical Magazine* for August 1838: another demonstration for the same case may be seen in the *Cambridge Mathematical Journal* for February 1839. I am not aware that up to the present time any purely algebraical demonstrations have been given for the cases of the ellipse and hyperbola. To supply this deficiency is the object of this paper. I shall first give a demonstration for the case of the hyperbola, and afterwards one for the ellipse, which, as will be seen, likewise includes another demonstration for the hyperbola.

I. The equation to the hyperbola referred to its asymptotes is

$$4xy = c^2.$$

Let the equation to a tangent-line be

$$\frac{x}{a_1} + \frac{y}{\beta_1} = 1:$$

then the roots of the equation

$$\frac{x^2}{a_1} - x + \frac{c^2}{4\beta_1} = 0,$$

which belong to the common point of the tangent and curve, must be equal: hence $a_1\beta_1 = c^2.$

Thus the equation to the tangent is

$$\frac{x}{a_1} + \frac{a_1 y}{c^2} = 1:$$

at the intersection of this tangent and another

$$\frac{x}{a_2} + \frac{a_2 y}{c^2} = 1,$$

we shall have $x_{1,2} = \dfrac{a_1 a_2}{a_1 + a_2}, \quad y_{1,2} = \dfrac{c^2}{a_1 + a_2}.$

These will be the co-ordinates of one of the angles of the circumscribed hexagon, the two tangents being two of the sides. We shall have analogous expressions for the co-ordinates of the other angles.

The equation to the diagonal through the angles $(x_{1,2}, y_{1,2})$, $(x_{4,5}, y_{4,5})$ will be

$$x(y_{4,5} - y_{1,2}) - y(x_{4,5} - x_{1,2}) = x_{1,2}\,y_{4,5} - x_{4,5}\,y_{1,2},$$

or, if we substitute for the co-ordinates of the angular points their values,

$$c^2x\left\{(a_4-a_1)-(a_2-a_5)\right\}+y\left\{a_4a_1(a_5-a_2)+a_5a_2(a_4-a_1)\right\}$$
$$=c^2(a_4a_5-a_1a_2)\ \dots\ (1).$$

Similarly the equation to the diagonal, through the angular points $(x_{2,3}, y_{2,3})$, $(x_{5,6}, y_{5,6})$, will be

$$c^2x\left\{(a_5-a_2)-(a_3-a_6)\right\}+y\left\{a_5a_2(a_6-a_3)+a_6a_3(a_5-a_2)\right\}$$
$$=c^2(a_5a_6-a_2a_3)\dots(2).$$

At the intersection of these two diagonals, multiplying the equation (1) by a_3-a_6, and the equation (2) by a_1-a_4, subtracting the latter of the resulting equations from the former, and dividing the final equation by a_2-a_5, we shall get

$$c^2x\left\{(a_6-a_3)-(a_4-a_1)\right\}+y\left\{a_6a_3(a_1-a_4)+a_1a_4(a_6-a_3)\right\}$$
$$=c^2(a_6a_1-a_3a_4)\dots(3).$$

But, as is evident from symmetry, equation (3) belongs to the third diagonal, namely, that which passes through the points $(x_{3,4}, y_{3,4})$, $(x_{6,1}, y_{6,1})$. Thus we see that the two diagonals (1) and (2) intersect in the third; which establishes the theorem.

Cor. Subtracting the sum of (1) and (3) from (2), we get for the value of y, at the point through which the three diagonals pass,

$$y\left\{a_1a_2(a_4+a_5)-a_2a_3(a_5+a_6)+a_3a_4(a_6+a_1)-a_4a_5(a_1+a_2)\right.$$
$$\left.+a_5a_6(a_2+a_3)-a_6a_1(a_3+a_4)\right\}=c^2(a_1a_2-a_2a_3+a_3a_4-a_4a_5+a_5a_6-a_6a_1).$$

The value of x, by virtue of symmetry, will be

$$x\left\{\beta_1\beta_2(\beta_4+\beta_5)-\beta_2\beta_3(\beta_5+\beta_6)+\beta_3\beta_4(\beta_6+\beta_1)-\beta_4\beta_5(\beta_1+\beta_2)\right.$$
$$\left.+\beta_5\beta_6(\beta_2+\beta_3)-\beta_6\beta_1(\beta_3+\beta_4)\right\}$$
$$=c^2(\beta_1\beta_2-\beta_2\beta_3+\beta_3\beta_4-\beta_4\beta_5+\beta_5\beta_6-\beta_6\beta_1).$$

II. The equation to the tangent of an ellipse is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1\ \dots\dots\dots\dots\ (1),$$

x' and y' being connected by the equation

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1.\dots\dots\dots\dots.(2).$$

Equation (2) is equivalent to the following system of equations,

$$\frac{x'}{a}+\frac{y'}{b}\sqrt{(-1)}=a_1,$$

$$\frac{x'}{a}-\frac{y'}{b}\sqrt{(-1)}=\frac{1}{a_1},$$

α_1 being an arbitrary quantity: hence

$$x' = a\left(\alpha_1 + \frac{1}{\alpha_1}\right), \quad y' = \frac{b}{\sqrt{(-1)}}\left(\alpha_1 - \frac{1}{\alpha_1}\right):$$

hence, equation (1) becomes

$$\frac{x}{a}\left(\alpha_1 + \frac{1}{\alpha_1}\right) + \frac{y}{b\sqrt{(-1)}}\left(\alpha_1 - \frac{1}{\alpha_1}\right) = 1.$$

Put $a = 2a'$, $b\sqrt{(-1)} = 2b'$; and the equation to the tangent, which we may suppose to be one of the sides of the hexagon, assumes the form

$$\frac{x}{2a'}\left(\alpha_1 + \frac{1}{\alpha_1}\right) + \frac{y}{2b'}\left(\alpha_1 - \frac{1}{\alpha_1}\right) = 1 \ldots\ldots\ldots (3).$$

The equation to the next side of the hexagon will be, in like manner, of the form

$$\frac{x}{2a'}\left(\alpha_2 + \frac{1}{\alpha_2}\right) + \frac{y}{2b'}\left(\alpha_2 - \frac{1}{\alpha_2}\right) = 1 \ldots\ldots (4).$$

At the intersection of (3) and (4), $x_{1,2}$, $y_{1,2}$ being the co-ordinates of the corresponding angle of the hexagon, we shall get, by the combination of the equations,

$$x_{1,2} = a'\frac{1 + \alpha_1\alpha_2}{\alpha_1 + \alpha_2}, \quad y_{1,2} = b'\frac{1 - \alpha_1\alpha_2}{\alpha_1 + \alpha_2}.$$

The equation to the diagonal, through the angular points $(x_{1,2}, y_{1,2})$, $(x_{4,5}, y_{4,5})$, will be

$$x(y_{4,5} - y_{1,2}) - y(x_{4,5} - x_{1,2}) = x_{1,2}y_{4,5} - x_{4,5}y_{1,2};$$

or, substituting for the co-ordinates of the angular points their values,

$$b'x\{(\alpha_1 - \alpha_4)(1 + \alpha_2\alpha_5) + (\alpha_2 - \alpha_5)(1 + \alpha_1\alpha_4)\} - a'y\{(\alpha_1 - \alpha_4)(1 - \alpha_2\alpha_5) \\ + (\alpha_2 - \alpha_5)(1 - \alpha_1\alpha_4)\} = 2a'b'(\alpha_1\alpha_2 - \alpha_4\alpha_5) \ldots (5).$$

By similarity, it is evident that the equation to the diagonal, through $(x_{2,3}, y_{2,3})$, $(x_{5,6}, y_{5,6})$, will be

$$b'x\{(\alpha_2 - \alpha_5)(1 + \alpha_3\alpha_6) + (\alpha_3 - \alpha_6)(1 + \alpha_2\alpha_5)\} - a'y\{(\alpha_2 - \alpha_5)(1 - \alpha_3\alpha_6) \\ + (\alpha_3 - \alpha_6)(1 - \alpha_2\alpha_5)\} = 2a'b'(\alpha_2\alpha_3 - \alpha_5\alpha_6) \ldots (6).$$

At the intersection of these two diagonals, multiplying (5) by $\alpha_3 - \alpha_6$, (6) by $\alpha_1 - \alpha_4$, subtracting the latter of the resulting equations from the former, and dividing the final equation by $\alpha_2 - \alpha_5$, we shall thus get

$$b'x\{(\alpha_3 - \alpha_6)(1 + \alpha_4\alpha_1) + (\alpha_4 - \alpha_1)(1 + \alpha_3\alpha_6)\} - a'y\{(\alpha_3 - \alpha_6)(1 - \alpha_4\alpha_1) \\ + (\alpha_4 - \alpha_1)(1 - \alpha_3\alpha_6)\} = 2a'b'(\alpha_3\alpha_4 - \alpha_6\alpha_1) \ldots (7).$$

But by the symmetry we know that (7) is the equation to the third diagonal; hence the diagonals (5), (6), intersect in (7).

III. The demonstration which we have given for the ellipse really comprehends a proof for the hyperbola. In fact the equation to the tangent at any point of an hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

may be expressed in the form

$$\frac{x}{2a'}\left(\alpha_1 + \frac{1}{\alpha_1}\right) + \frac{y}{2b'}\left(\alpha_1 - \frac{1}{\alpha_1}\right) = 1,$$

by the process given for the ellipse. In the case of the hyperbola $2a' = a$, $2b' = -b$. Thus the demonstration for the ellipse coincides with that for the hyperbola.

V.—NOTES ON LINEAR TRANSFORMATIONS.

By George Boole.

1. The complete solution of the problem of which the object is to take away the products of the variables from a homogeneous function of the second degree,

$$ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy \ldots\ldots (1),$$

requires the determination of the coefficients α, β, γ, &c. in the linear theorems

$$x = \alpha x' + \beta y' + \gamma z' \ldots\ldots\ldots\ldots (2),$$
$$y = \alpha' x' + \beta' y' + \gamma' z' \ldots\ldots\ldots\ldots (3),$$
$$z = \alpha'' x' + \beta'' y' + \gamma'' z' \ldots\ldots\ldots\ldots (4),$$

as well as of a', b', c', in the transformed function

$$a'x'^2 + b'y'^2 + c'z'^2 \ldots\ldots\ldots\ldots (5).$$

The following investigation is intended to effect this object.

The transformation being supposed to be from one rectangular system of co-ordinates to another, we shall have, on squaring (2), the following system of equations of the second degree,

$$\left.\begin{aligned} x^2 &= \alpha^2x'^2 + \beta^2y'^2 + \gamma^2z'^2 + 2\beta\gamma y'z' + 2\gamma\alpha z'x' + 2\alpha\beta x'y' \\ x^2 + y^2 + z^2 &= x'^2 + y'^2 + z'^2 \\ ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy &= a'x'^2 + b'y'^2 + c'z'^2 \end{aligned}\right\}\ldots(6).$$

Treating this system of equations by the method given in pp. 108–114 of the *Mathematical Journal*, we obtain six

equations among the constants. Three of these equations are included in the cubic

$$(\eta-a)(\eta-b)(\eta-c)-d(\eta-a)-e(\eta-b)-f(\eta-c)+2def=0,$$

in which the values of η determine a', b', c'. The other three equations are

$$\alpha^2+\beta^2+\gamma^2=1 \dots\dots\dots\dots\dots (7),$$

$$(b'+c')\alpha^2+(c'+a')\beta^2+(a'+b')\gamma^2=b+c \dots (8),$$

$$b'c'\alpha^2+c'a'\beta^2+a'b'\gamma^2=bc-d^2 \dots\dots\dots (9).$$

Hence $(7)\times a'^2-(8)\times a'+(9)$ gives

$$(a'^2+b'c'-a'b'-a'c')\alpha^2=a'^2-(b+c)a'+bc-d^2,$$

$$\therefore \alpha^2=\frac{(a'-b)(a'-c)-d^2}{(a'-b')(a'-c')};$$

whence, by inspection,

$$\beta^2=\frac{(b'-b)(b'-c)-d^2}{(b'-c')(b'-a')}, \quad \gamma^2=\frac{(c'-b)(c'-c)-d^2}{(c'-a')(c'-b')}.$$

To obtain $\alpha'^2, \beta'^2, \gamma'^2$, we must change in the above, b, c, d into c, a, e, respectively; and to find $\alpha''^2, \beta''^2, \gamma''^2$, we must change b, c, d into a, b, f, respectively.

2. *Attraction of an Ellipsoid.* In investigating the attraction of the ellipsoid whose surface is defined by the equation

$$\frac{x^2}{h^2}+\frac{y^2}{h_1^2}+\frac{z^2}{h_2^2}=1$$

on an external point, a, b, c, the force varying as $\frac{1}{d^2}$, we meet with an equation of the form

$$\frac{a^2}{\eta^2+h^2}+\frac{b^2}{\eta^2+h_1^2}+\frac{c^2}{\eta^2+h_2^2}=1.$$

It may be worth while to observe that the values of η^2, as determined by the above, are respectively equal to those which we should find for A, B, C, in transforming the homogeneous function

$$(a^2-h^2)x^2+(b^2-h_1^2)y^2+(c^2-h_2^2)z^2+2bcyz+2cezx+2abxy$$

into the form
$$Ax'^2+By'^2+Cz'^2,$$

x, y, z, and x', y', z', denoting rectangular systems of co-ordinates.

3. THEOREM. *If Q be a homogeneous function of the n^{th} degree with m variables, $x_1, x_2 \dots x_m$, which is transformed into*

R, a similar homogeneous function, by the relations

$$\left.\begin{array}{l} x_1 = \lambda_1 y_1 + \lambda_2 y_2 \dots + \lambda_m y_m \\ \dots\dots\dots\dots\dots\dots \\ x_m = \rho_1 y_1 + \rho_2 y_2 \dots + \rho_m y_m \end{array}\right\} \dots\dots\dots\dots (1);$$

and if γ be the degree of $\theta(Q)$ and $\theta(R)$, then

$$\theta(Q) = \frac{\theta(R)}{E^{\frac{\gamma n}{m}}} \dots\dots\dots\dots\dots (2),$$

where E is the result obtained by eliminating the variables from the second members of (1) *equated to* 0.

The above theorem was given, but without demonstration, in vol. III. p. 19 of the *Mathematical Journal*. The following is the analysis by which it was obtained.

Let $q = r$ represent another equation analogous to the equation $Q = R$, q being a homogeneous function of $x_1 .. x_m$ of the n^{th} degree, r a similar function of $y_1, y_2 .. y_m$. By the theory of linear transformations (vol. III. p. 9), the equations $\theta(Q + hq) = 0$, $\theta(R + hr) = 0$, are identical relatively to h. If we suppose $\theta(Q + hq)$ expanded in ascending powers of h, we shall have an equation of the form

$$L + Mh \dots + Zh^{\gamma} = 0,$$

wherein $L = \theta(Q)$, and Z, the last coefficient, $= \theta(q)$. In like manner $\theta(R + hr) = 0$ will assume the form

$$L_1 + M_1 h \dots + Z_1 h^{\gamma} = 0,$$

in which $L_1 = \theta(R)$, $Z_1 = \theta(r)$. That the two equations may give identical values of h, a series of conditions must be fulfilled, of which the last is

$$\frac{L}{Z} = \frac{L_1}{Z_1},$$

wherefore

$$\frac{\theta(Q)}{\theta(q)} = \frac{\theta(R)}{\theta(r)} \dots\dots\dots\dots\dots (3).$$

Now let $x_1, x_2 .. x_m$ vanish, then (1) gives

$$\left.\begin{array}{r} \lambda_1 y_1 .. + \lambda_m y_m = 0 \\ \dots\dots\dots\dots \\ \rho_1 y_1 .. + \rho_m y_m = 0 \end{array}\right\} \dots\dots\dots\dots (4).$$

Eliminating the variables, we have $E = 0$, which is the necessary and sufficient condition, that $x_1 \dots x_m$ may vanish without causing $y_1 \dots y_m$ to vanish.

Now, vol. III. p. 4, the condition

$$\frac{dq}{dx_1}=0,\quad \frac{dq}{dx_2}=0\ \ldots\ \frac{dq}{dx_m}=0\ \ldots\ldots\ldots\ (5)$$

induces as a necessary consequence the condition

$$\frac{dr}{dy_1}=0,\quad \frac{dr}{dy_2}=0\ \ldots\ \frac{dr}{dy_m}=0\ \ldots\ldots\ldots\ (6).$$

Assume $q=x_1^n+x_2^n\ldots+x_m^n$, then

$$\frac{dq}{dx_1}=nx_1^{n-1}\ \ldots\ \frac{dq}{dx_m}=nx_m^{n-1}.$$

Hence, if $x_1, x_2 \ldots x_m$, vanish, the condition (5) is satisfied, and therefore (6). Eliminating the variables from (6), we have $\theta(r)=0$, the necessary and sufficient condition in order that $x_1, x_2 \ldots x_m$, may vanish without causing $y_1, y_2 \ldots y_m$, to vanish also.

Since, from the above, $\theta(r)$ and E are so related that the one cannot vanish without causing the other to vanish also, they must be connected by an equation of the form

$$\theta(r)=CE^{\lambda}\ \ldots\ldots\ldots\ldots\ldots\ (7),$$

C being a constant, and λ a positive constant to be determined. Both C and λ, it is to be observed, are quite independent of $\lambda_1 \ldots \lambda_m$, $\rho_1 \ldots \rho_m$, the constants in the linear theorems.

If in q we substitute for $x_1, x_2 \ldots x_m$, their values from (1) in terms of $y_1, y_2 \ldots y_m$, the resulting coefficients of r will each be of the n^{th} degree in terms of λ_1, ρ_1, &c.; wherefore $\theta(r)$ is of the $(\gamma n)^{\text{th}}$ degree with respect to $\lambda_1 \ldots \rho_1$, &c. Now E is of the m^{th} degree with respect to those quantities, wherefore $\theta(r)$ is of the $\left(\gamma\frac{n}{m}\right)^{\text{th}}$ degree with respect to E, and λ in (7) $=\gamma\frac{n}{m}$.

As C is quite independent of $\lambda_1 \ldots \rho_1$, &c., we may, in determining C, attribute to those quantities what values we please. Assume them such that we may have

$$x_1=y_1,\quad x_2=y_2\ \ldots\ x_m=y_m,$$

and let r_1 be the particular value of r under these circumstances, then $\theta(r_1)=\theta(q)$, $E=1$, wherefore, by (7),

$$C=\theta(r_1)=\theta(q),$$

whence, substituting in (7) for C and λ,

$$\theta(r)=\theta(q)\,E^{\frac{\gamma n}{m}}.$$

Employing this value of $\theta(r)$ in (3), we have

$$\theta(Q) = \frac{\theta(R)}{E^{\frac{\gamma n}{m}}}.$$

From a formula given in one of Professor Sylvester's papers on Elimination, in the *Philosophical Magazine*, it appears, that $\gamma = m(n-1)^{m-1}$, so that the above would give

$$\theta(Q) = \frac{\theta(R)}{E^{n(n-1)^{m-1}}}.$$

4. *Determination of $\theta(Q)$ when Q is a homogeneous function of the fourth degree with two variables.*

When Q is of the form $ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$, the value of $\theta(Q)$ is found by eliminating the variables from the equations

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0,$$

$$bx^3 + 3cx^2y + 3dxy^2 + ey^3 = 0.$$

After a tedious process, we find

$$\begin{aligned}\theta(Q) = {} & a^3e^3 - 6ab^2d^2e - 12a^2bde^2 - 18a^2c^2e^2 - 27a^2d^4 - 27b^4e^2 \\ & + 36b^2c^2d^2 + 54a^2cd^2e + 54ab^2ce^2 - 54ac^3d^2 - 54b^2c^3e - 64b^3d^3 \\ & + 81ac^4e + 108abcd^3 + 108b^3cde - 180abc^2de.\end{aligned}$$

The above result may probably be found to have some applications in the theory of the higher algebraic equations.

Lincoln, June, 1844.

VI.—ON A PROBLEM IN CENTRAL FORCES.

A PARTICLE moves about a centre of attractive force varying directly as the distance; to determine the motion, having given the velocity and direction of projection, and also the initial position of the particle.

The solution of this problem is ordinarily effected, either by means of the polar differential equation, or by resolving the force in directions parallel to two rectangular axes. The motion however may be more conveniently referred to a pair of oblique axes, selected as we shall explain in this paper.

Let the centre of force be taken as the origin of co-ordinates, and let the axis of x be chosen so as to pass through the initial position of the particle. Let the axis of y be taken parallel to the direction of projection. The co-ordinate axes will thus generally be oblique to each other.

For the motion there is, μ^2 denoting the absolute force,

$$\frac{d^2x}{dt^2} + \mu^2 x = 0 \ldots\ldots\ldots\ldots (1),$$

$$\frac{d^2y}{dt^2} + \mu^2 y = 0 \ldots\ldots\ldots\ldots (2).$$

The integral of equation (1) is

$$x = A \cos(\mu t + \varepsilon),$$

A, ε, being arbitrary constants. Let a be the initial value of x; then

$$a = A \cos \varepsilon;$$

and, since $\frac{dx}{dt}$ is initially equal to zero,

$$0 = -A\mu \sin \varepsilon;$$

hence the integral becomes

$$x = a \cos(\mu t) \ldots\ldots\ldots\ldots (3).$$

The integral of (2) is

$$y = A' \cos(\mu t + \varepsilon'),$$

A', ε', being arbitrary constants. Let v be the initial value of $\frac{dy}{dt}$; then, zero being the initial value of y,

$$0 = A' \cos \varepsilon',$$

$$v = -A' \mu \sin \varepsilon',$$

and therefore the integral becomes

$$y = \frac{v}{\mu} \sin(\mu t) \ldots\ldots\ldots\ldots (4).$$

From (3) and (4) we see that, putting $\frac{v}{\mu} = b$,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right) = 1 \ldots\ldots\ldots\ldots (5),$$

which is the equation to an ellipse of which the centre coincides with the centre of force, the directions of x and y coinciding with the semi-conjugate diameters a, b.

From (3) and (4) we see that the values of x, y, $\frac{dx}{dt}$, $\frac{dy}{dt}$, will not be altered if t be so increased that μt increase by 2π; hence it follows that the periodic time is equal to $\frac{2\pi}{\mu}$.

Let r be the distance of any point in the path from the origin; then, ω being the angle between the axes,

$$r^2 = x^2 + y^2 + 2xy \cos \omega \ldots\ldots\ldots (6).$$

At the extremities of the axes of the ellipse, $dr = 0$; hence, for the determination of these points, we have, from (5) and (6),

$$0 = \frac{xdx}{a^2} + \frac{ydy}{b^2},$$

$$0 = (x + y \cos \omega)\, dx + (y + x \cos \omega)\, dy;$$

whence, λ being an arbitrary quantity,

$$\left(\frac{\lambda}{a^2} - 1\right) x = y \cos \omega \ldots\ldots\ldots\ldots (7),$$

$$\left(\frac{\lambda}{b^2} - 1\right) y = x \cos \omega \ldots\ldots\ldots\ldots (8).$$

From (7) and (8) there is, multiplying them by x, y, respectively, adding, and attending to (5) and (6),

$$\lambda = r^2 \ldots\ldots\ldots\ldots\ldots\ldots (9):$$

also, multiplying together (7) and (8), and dividing by xy, we have, by virtue of (9),

$$\left(\frac{r^2}{a^2} - 1\right)\left(\frac{r^2}{b^2} - 1\right) = \cos^2 \omega \ldots\ldots\ldots (10),$$

which gives two values of r^2, the one belonging to the semi-axis major, and the other to the semi-axis minor.

The equations to the semi-axes, r', r'', denoting the two roots of (10), are

$$\left(\frac{r'^2}{a^2} - 1\right) x = y \cos \omega,$$

$$\left(\frac{r''^2}{a^2} - 1\right) x = y \cos \omega:$$

or, which, by virtue of (10), comes to the same thing,

$$\left(\frac{r'^2}{b^2} - 1\right) y = x \cos \omega,$$

$$\left(\frac{r''^2}{b^2} - 1\right) y = x \cos \omega.$$

W. W.

VII.—NOTE ON THE SPONTANEOUS AXIS OF ROTATION.

IN Article III. of this number of the *Journal*, on the Spontaneous Axis of Rotation, it is shewn that, under certain circumstances, expressed analytically by the equation (8), if

a free rigid body be struck by impulsive forces, there will be a series of particles of the body in a straight line, which *ipso motûs initio* enjoy absolute rest. This article recals my attention to certain researches in which I had been engaged in connection with the same problem. The conclusions at which I had arrived, although, as far as they went, in harmony with the results of the article to which I have alluded, are not however precisely the same, in consequence of a different definition of the Spontaneous Axis. I had adopted this term to denote the rectilinear locus of a line of particles within the body, all of which, on the application of the impulsive forces, assume a velocity in the direction of the line itself. Thus, according to this definition, a cone struck so as to descend with its axis vertical, whatever be the rotatory motion of the cone, will have its axis of figure for its spontaneous axis. Considering V_x, V_y, V_z: constant in equations (6) of Art. III., any two of these equations will represent a straight line. Multiplying them in order by $\frac{L}{A}$, $\frac{M}{B}$, $\frac{N}{C}$, we get, as the condition for their coexistence,

$$\frac{L.V_x}{A} + \frac{M.V_y}{B} + \frac{N.V_z}{C} = \frac{LX}{mA} + \frac{MY}{mB} + \frac{NZ}{mC} \dots\dots (\alpha).$$

The direction-cosines of the line are, as appears from the equations to the line, proportional to

$$\frac{L}{A}, \ \frac{M}{B}, \ \frac{N}{C}:$$

but, by the definition of the spontaneous axis, these cosines must be also proportional to

$$V_x, V_y, V_z:$$

hence, putting

$$\left.\begin{aligned} V_x &= \frac{kL}{A} \\ V_y &= \frac{kM}{B} \\ V_z &= \frac{kN}{C} \end{aligned}\right\} \dots\dots\dots\dots\dots (\beta),$$

we see from (α), that

$$k = \frac{\dfrac{LX}{mA} + \dfrac{MY}{mB} + \dfrac{NZ}{mC}}{\dfrac{L^2}{A^2} + \dfrac{M^2}{B^2} + \dfrac{N^2}{C^2}} \dots\dots\dots\dots (\gamma).$$

From (β) and (γ), V denoting the velocity of the spontaneous axis, we see that

$$V = (V_x^2 + V_y^2 + V_z^2)^{\frac{1}{2}}$$
$$= k\left(\frac{L^2}{A^2} + \frac{M^2}{B^2} + \frac{N^2}{C^2}\right)^{\frac{1}{2}}$$
$$= \frac{\frac{LX}{mA} + \frac{MY}{mB} + \frac{NZ}{mC}}{\left(\frac{L^2}{A^2} + \frac{M^2}{B^2} + \frac{N^2}{C^2}\right)^{\frac{1}{2}}}.$$

The equations will then become

$$\left.\begin{aligned} \frac{kL}{A} &= \frac{X}{m} + z\frac{M}{B} - y\frac{N}{C} \\ \frac{kM}{B} &= \frac{Y}{m} + x\frac{N}{C} - z\frac{L}{A} \\ \frac{kN}{C} &= \frac{Z}{m} + y\frac{L}{A} - x\frac{M}{B} \end{aligned}\right\} \dots\dots\dots\dots \delta),$$

which are the equations to the spontaneous axis, k being supposed to have the value given in the equation (γ).

Suppose that, for the locus of certain points, V_x, V_y, V_z, are constant, without being subject to any other condition except that given by the equations (α). Equations (6) will then represent a straight line parallel to line (δ); and, since V_x, V_y, V_z, may receive an infinite variety of values without violating the essential condition (α), it appears that there is an infinite number of such straight lines in the body.

The locus of those points of which the velocities, irrespectively of their directions, have the same value V, will be a cylinder, of which the equation is

$$V^2 = \left(\frac{X}{m} + z\frac{M}{B} - y\frac{N}{C}\right)^2$$
$$+ \left(\frac{Y}{m} + x\frac{N}{C} - z\frac{L}{A}\right)^2$$
$$+ \left(\frac{Z}{m} + y\frac{L}{A} - x\frac{M}{B}\right)^2,$$

the number of such surfaces being infinite in accordance with the variations of the value of V and their common axis being the line (δ).

W. W.

VIII.—APPLICATIONS OF THE SYMBOLICAL FORM OF MACLAURIN'S THEOREM.

The symbolical form of Maclaurin's Theorem may sometimes be applied with advantage to the demonstration of algebraical and trigonometrical formulæ consisting of a finite number of terms. I shall give two instances of its use in such cases. It seems probable that the application of this form of the theorem might be extended to the investigation of the properties of various infinite series in trigonometry: the fundamental principles, however, of its application to such purposes would stand in need of examination.

1. To prove the polynomial theorem.

By Maclaurin's theorem, there is

$$(a + a' + a'' + \dots)^n = \varepsilon^{a\frac{d}{d0}} \cdot \varepsilon^{a'\frac{d}{d0'}} \cdot \varepsilon^{a''\frac{d}{d0''}} \cdots (0 + 0' + 0'' + \dots)^n,$$

of which the general term is evidently

$$\frac{a^r . a'^{r'} . a''^{r''} \dots}{1 \dots r . 1 \dots r' . 1 \dots r'' \dots} \cdot \left(\frac{d}{d0}\right)^r \left(\frac{d}{d0'}\right)^{r'} \left(\frac{d}{d0''}\right)^{r''} \dots (0 + 0' + 0'' + \dots)^n.$$

Now $\left(\frac{d}{d0}\right)^r \left(\frac{d}{d0'}\right)^{r'} \left(\frac{d}{d0''}\right)^{r''} \cdots (0 + 0' + 0'' + \dots)^n$

$$= \left(\frac{d}{d0}\right)^{r+r'+r''+} \quad 0^n$$

$$= n(n-1)(n-2)\dots\{n - (r + r' + r'' + \dots) + 1\}\ 0^{n-r-r'-r''\dots}$$

which, supposing n to be a positive integer, is always equal to zero, except when

$$n - r - r' - r'' - \dots = 0,$$

in which case it becomes

$$1.2.3.\dots n.$$

Hence, if n be a positive integer, the general term of $(a + a' + a'' + \dots)^n$ is

$$\frac{1.2.3.\dots n}{1.2\dots r \times 1.2\dots r' \times 1.2\dots r'' \times \dots}\ a^r\, a'^{r'}\, a''^{r''} \dots$$

under the condition

$$r + r' + r'' + \dots = n.$$

2. To prove the formula for the development of $(\cos\theta)^n$ by cosines of multiple angles, when n is a positive integer.

Since $2\cos\theta = \cos\theta + \cos(-\theta)$, it is clear that

$$2\cos\theta = (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}}) \cos 0,$$

$$(2\cos\theta)^2 = (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})\, 2\cos 0\cos\theta$$

$$= (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})\,\{\cos(0+\theta) + \cos(0-\theta)\}$$

$$= (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})(\varepsilon^{\theta\frac{d}{d0'}} + \varepsilon^{-\theta\frac{d}{d0'}})\cos(0+0')$$

$$= (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})(\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})\cos(0+0'),$$

since $\dfrac{d}{d0'}$, operating on a function of $0+0'$, produces the same result as $\dfrac{d}{d0}$, $\quad = (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})^2\cos 0.$

Proceeding in the same way, it is obvious that we should get

$$(2\cos\theta)^n = (\varepsilon^{\theta\frac{d}{d0}} + \varepsilon^{-\theta\frac{d}{d0}})^n\cos 0$$

$$= \Big\{\varepsilon^{n\theta\frac{d}{d0}} + n\,\varepsilon^{(n-2)\theta\frac{d}{d0}} + \frac{n(n-1)}{1.2}\,\varepsilon^{(n-4)\theta\frac{d}{d0}} + \dots\dots$$

$$+ \frac{n(n-1)}{1.2}\,\varepsilon^{-(n-4)\theta\frac{d}{d0}} + n\varepsilon^{-(n-2)\theta\frac{d}{d0}} + \varepsilon^{-n\theta\frac{d}{d0}}\Big\}\cos 0$$

$$= \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1.2}\cos(n-4)\theta + \dots\dots$$

$$+ \frac{n(n-1)}{1.2}\cos\{-(n-4)\theta\} + n\cos\{-(n-2)\theta\} + \cos(-n\theta).$$

3. Putting $\frac{1}{2}\pi - \theta$ for θ in the formula for $(2\cos\theta)^n$, we have

$$(2\sin\theta)^n = \cos\{n(\tfrac{1}{2}\pi-\theta)\} + n\cos\{(n-2)(\tfrac{1}{2}\pi-\theta)\}$$

$$+ \frac{n(n-1)}{1.2}\cos\{(n-4)(\tfrac{1}{2}\pi-\theta)\} + \dots$$

$$+ \frac{n(n-1)}{1.2}\cos\{-(n-4)(\tfrac{1}{2}\pi-\theta)\}$$

$$+ n\cos\{-(n-2)(\tfrac{1}{2}\pi-\theta)\}$$

$$+ \cos\{-n(\tfrac{1}{2}\pi-\theta)\},$$

a general formula comprehending the four cases ordinarily considered separately, accordingly as n is of the form $4m$, $4m+1$, $4m+2$, or $4m+3$.

W. W.

IX.—ON THE USE OF THE SYMBOL $e^{\theta\sqrt{(-1)}}$ IN CERTAIN TRANSFORMATIONS.

THE symbol $e^{\theta\sqrt{(-1)}}$, considered as a sign of affection determining the direction in which a straight line is drawn, may be

successfully applied to effect several transformations from rectangular to polar co-ordinates: and the application of the symbol to this purpose will perhaps be useful, not only for the sake of the transformations themselves, but also in the light of illustrating generally the meaning of the symbol $e^{\theta\sqrt{(-1)}}$.

Let it be required to transform the element $dx\,dy$ to polar co-ordinates. We have

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$\therefore\quad x + y\sqrt{(-1)} = r\{\cos\theta + \sqrt{(-1)}\sin\theta\} = re^{\theta\sqrt{(-1)}};$$

differentiating,

$$dx + dy\sqrt{(-1)} = e^{\theta\sqrt{(-1)}}\{dr + rd\theta\sqrt{(-1)}\}\ldots\ldots(1).$$

Now the right hand side of the equation is similar to the left, with the exception of being multiplied by $e^{\theta\sqrt{(-1)}}$, which being only a symbol of direction, need not be considered when the question is merely one of magnitude; therefore, equating possible and impossible parts,

$$\left.\begin{aligned} dx &= dr \\ dy &= rd\theta \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$
$$\therefore\quad dx\,dy = rd\theta\,dr,$$

which is the transformation required.

We may also reason thus: the sign of affection $e^{\theta\sqrt{(-1)}}$ merely signifies that the line along which r is measured is inclined at an angle θ to the axis of x; hence, after the differentiation has been performed, we may make $\theta = 0$, or suppose the axis of x to coincide with the direction of r, and thus $dx + dy\sqrt{(-1)} = dr + rd\theta\sqrt{(-1)}$, and equations (2) follow as before.

Again, let it be required to find the effective accelerating forces in the direction of and perpendicular to the radius vector, when the motion is in one plane. We have

$$dx + dy\sqrt{(-1)} = e^{\theta\sqrt{(-1)}}\{dr + rd\theta\sqrt{(-1)}\}$$
$$d^2x + d^2y\sqrt{(-1)} = e^{\theta\sqrt{(-1)}}\{d^2r - rd\theta^2 + \sqrt{(-1)}(rd^2\theta + 2drd\theta)\}\ldots(3);$$

and therefore, from the same reasoning as before,

$$\frac{d^2r}{dt^2} - r\frac{d\theta^2}{dt^2} = \frac{d^2x}{dt^2} = \text{effective accelerating}$$

force in the direction of r,

$$r\frac{d\theta^2}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{d^2y}{dt^2} = \text{effective}$$

accelerating force perpendicular to the direction of r.

The preceding formulæ will enable us to treat very neatly the equations of motion of a disturbed planet in two dimensions. (*Airy's Tracts*, p. 64.)

For we have
$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{dR}{dx} = 0 \\ \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{dR}{dy} = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (4),$$

$$\therefore \frac{d^2x + \sqrt{(-1)}\, d^2y}{dt^2} + \frac{\mu}{r^3}\{x + y\sqrt{(-1)}\} + \frac{dR}{dx} + \sqrt{(-1)}\,\frac{dR}{dy} = 0,$$

or $$\frac{d^2r}{dt^2} - r\,\frac{d\theta^2}{dt^2} + \sqrt{(-1)}.\,\frac{1}{r}\,\frac{d}{dt}\left(r^2\,\frac{d\theta}{dt}\right) + \frac{\mu}{r^2} + \frac{dR}{dr} + \sqrt{(-1)}\,\frac{dR}{rd\theta} = 0,$$

(where $e^{\theta\sqrt{(-1)}}$ has been put = 1);

$$\left.\begin{aligned} \therefore \frac{d^2r}{dt^2} - r\,\frac{d\theta^2}{dt^2} + \frac{\mu}{r^2} + \frac{dR}{dr} = 0 \\ \frac{d}{dt}\left(r^2\,\frac{d\theta}{dt}\right) + \frac{dR}{d\theta} = 0 \end{aligned}\right\} \ldots\ldots\ldots (5),$$

which are the equations required.

The same method will apply to other transformations.

H. G.

X.—NOTE ON ORTHOGONAL ISOTHERMAL SURFACES.

In a previous paper in this *Journal* ("On the Equations of the Motion of Heat referred to Curvilinear Co-ordinates," vol. IV. p. 33), I expressed the conditions which must be satisfied by a system of conjugate orthogonal surfaces which are all isothermal, and considered some particular cases of such systems. In addition to those, however, there is another class of surfaces which satisfy the conditions. For it is readily seen that all that is necessary in the demonstration of the theorem relative to cylindrical surfaces in p. 38, is that H_2 shall be independent of λ and λ_1, and that $\dfrac{H_1}{H}$ shall be independent of λ_2. Now a series of concentric spheres (including the case of a series of parallel planes) is such that the value of H_2 at any point of one of them is independent of the position of the point on that surface. Hence we may consider $\lambda_2 = a_2$ as representing a series of concentric spheres, and consequently $\lambda = a$, and $\lambda_1 = a_1$, a system of conjugate orthogonal cones having the centre of the spheres for their common

vertex. Any cone of one series will intersect any one of the conjugate series along a generating line, and the values of H and H_1 at points along this line will vary as the distances from the centre. Hence $\frac{H_1}{H}$ is independent of λ_2. We therefore see that the demonstration in p. 38 is applicable to the general class of orthogonal cones, as well as to the particular included case of orthogonal cylinders. Thus we have the general theorem that, if a series of cones having a common vertex be isothermal, the series of orthogonal cones will also be isothermal.

This class, and the class of confocal surfaces of the second degree, are all the triple systems which as yet have been found to be isothermal; and, in the paper already referred to, some particular cases were considered in which it was shown that one, or two of the partial series of an orthogonal system are isothermal, and the remaining series not. Since the publication of that paper I have received a Mémoire by M. Lamé, which was published about the same time (*Journal de Mathématiques*, vol. VIII. p. 397, Oct. and Nov. 1843), in which he shows that no other triple isothermal system can exist. This interesting result is the complete answer of a question proposed in this *Journal*, in May 1843 (vol. III. p. 286), and to which a partial answer was given in the paper already referred to (Nov. 1843). The same question has been proposed by M. Bertrand, in the April number of the *Journal de Mathématiques* of the present year (vol. IX. p. 117); and he answers it to a similar extent, by showing that an isothermal series of surfaces has not in every case its two conjugate orthogonal series isothermal also. The reason, however, which he assigns for coming to this conclusion does not seem to be quite satisfactory: for it is founded on the assumption that "it is always possible to take two consecutive isothermal surfaces arbitrarily, and that the law of the temperature of the rest of the body is then determined." Now, by considerations analogous to those brought forward in a paper "On some Points in the Theory of Heat" in this *Journal* (vol. IV. p. 71), it is readily seen that if two consecutive isothermal surfaces be arbitrarily assumed, it will in general be only for *points between them* that a possible system of isothermal surfaces can be determined according to a continuous law. The temperatures of points not lying between them will follow a different law depending on the sources of heat or cold which we must suppose to be distributed over the two assumed surfaces, to retain them at their constant temperatures. Thus, if we

assume arbitrarily two consecutive isothermal surfaces indefinitely near one another, the system of isothermal surfaces through the whole body, to which these two belong, will in general be impossible. In fact, it will generally be impossible to find any two surfaces, containing the two assumed ones between them, which will be such that if they be retained at different constant temperatures, the two assumed surfaces will each be isothermal. But M. Bertrand's conclusion, though correct, is drawn from the assumption that this is generally possible. It may be remarked, however, that though some restriction is necessary in assuming two consecutive surfaces of a possible isothermal system, it will probably be found to be not so narrow as a restriction which M. Bertrand shows to be necessary in choosing two consecutive surfaces of an isothermal series of which the conjugate orthogonal series are isothermal also. If this were previously shown, M. Bertrand's inference would be correct.

M. Bertrand also specially considers the case of isothermal orthogonal surfaces of revolution, and arrives at the interesting theorem that, if each of the conjugate series be isothermal, the traces on the meridian planes will form a system of conjugate isothermal plane curves, or the traces of a system of conjugate isothermal cylinders, on their orthogonal planes.

This follows at once from the equations (14) and (15) (vol. IV. p. 39 of this *Journal*), though it did not occur to me till I saw M. Bertrand's paper. For, from them we deduce

$$\frac{H_1}{H} = \phi(\lambda) \cdot \phi(\lambda_1),$$

which is the sole condition that each of the series of orthogonal plane curves represented by $\lambda = a$, $\lambda_1 = a_1$, shall be isothermal.

Hence we see that if this condition be satisfied, and at the same time the condition expressed by equation (16), the two conjugate orthogonal series of surfaces of revolution will also be isothermal. M. Bertrand states the latter condition in geometrical language as follows—

"Two systems of isothermal orthogonal lines being given, in order that their rotation round an axis may generate isothermal surfaces of revolution, it is necessary that the distances from the four corners of a curvilinear rectangle formed by the given lines shall be the four terms of an analogy."

We may also add, that if a single series of surfaces of revolution be isothermal, and if the traces on a meridian

plane be isothermal lines, then the conjugate orthogonal series of surfaces will also be isothermal.

Also it follows, from the result of M. Lamé's investigations mentioned above, that confocal surfaces of revolution of the second order form the only isothermal system which trace a series of isothermal lines on a meridian plane.

P. Q. R.

XI.—ON THE SOLUTION OF EQUATIONS IN FINITE DIFFERENCES.

By R. L. Ellis, M.A. Fellow of Trinity College.

The partial differential equations which occur in various branches of mathematical physics are, for the most part, of such forms that solutions of them may be obtained without much difficulty. As is well known, the great difficulty in almost all such cases consists in the necessity of determining which of all possible solutions satisfies the particular conditions of the problem on which we are engaged. It seems that before the time of Fourier's researches on heat, the course which mathematicians had uniformly followed was, first to obtain the general solution of the equation of the problem, and then to determine by particular considerations the arbitrary functions which it involved. This course undoubtedly would be the most direct and analytical, were there any general method for determining the form of the functions in question: as, however, there is none, the analytical generality of the first part of the process is in many cases sterile and useless.

Fourier's methods, which depend essentially on the linearity of the partial differential equations which occur in the theory of heat, consist in assuming some simple solution of the equation of the problem, in deducing from hence a more general solution of it, and in determining successively and by means of particular considerations the arbitrary quantities thus introduced in such a manner as to satisfy all the conditions of the question. The general solution with arbitrary functions does not make its appearance in his process; and the reason why it is so much more manageable than the other appears to be, that it is far easier to determine arbitrary constants in accordance with certain conditions than arbitrary functions. There will, generally speaking, be an infinite number of arbitrary constants, and it is therefore necessary to treat them in *classes*. The ingenious synthesis by which this is effected by Fourier, in the different problems discussed by him in the *Théorie de la Chaleur*, forms one of the most interesting parts of that admirable work. The same kind of

reasoning is made use of by Poisson, in his researches on similar subjects: and there can be little doubt that the methods of Fourier, developed and extended as they have been by subsequent writers, will long continue to be an essential element in the application of mathematics to physical researches. Similar methods may be made available in the solution of equations in partial finite differences. Such equations do not, it is true, present themselves very often, as the continuity of the causes to which natural phenomena are due, leads rather to differential equations than to those in finite differences. In fact, I am not aware of any subject, except the theory of probabilities, in which we meet with problems whose solution depends on that of an equation in partial finite differences.

In this theory, however, such problems are not uncommon. One of the most interesting of them, both in its own nature and historically, may serve as an illustration of the application of the methods of Fourier to finite differences. This problem, which has engaged the attention of several writers on the subject of probabilities, and of which a solution was among the earliest efforts of Ampère, is that of the *duration of play*. Professor De Morgan has spoken of this solution and of that of Laplace, as being of the highest order of difficulty: that which I am about to enter on has, I think, a decided advantage in this respect.

The problem itself may be thus stated:—Two persons, M and N, have between them a number a of counters: they play at a game at which M's chance is p, and N's q. The losing player gives one counter to the other, and they are to play on until one or other have lost all his counters. What is the probability that the party will terminate in M's favour after any assigned number of games, N being supposed to have originally x of the a counters?

Let y_{xz} be the probability that M will win the party at the $(z+1)^{\text{th}}$ game. If he win the next game (of which the probability is p), this becomes $y_{x-1\cdot z-1}$; if he lose it (of which the probability is q), it becomes $y_{x+1\cdot z-1}$, and therefore

$$y_{xz} = py_{x-1\cdot z-1} + qy_{x+1\cdot z-1} \dots\dots\dots\dots (1).$$

This is the equation of the problem. It is clear that

$$y_{0z} = 0, \quad y_{az} = 0 \dots\dots\dots\dots (2),$$

as the party ceases as soon as M or N has a counters. Again,

$$y_{x0} = 0 \text{ unless } x = 1, \text{ and } y_{1\cdot 0} = p \dots\dots (3);$$

for if N have more than one counter he cannot lose them all

at the next game; and if he have only one, his chance of his being left without any is p.

Let us assume $$y_{xz} = \alpha^z \nu_x \ldots\ldots\ldots\ldots\ldots\ldots (4),$$
α being arbitrary. Then
$$\alpha\nu_x = p\nu_{x-1} + q\nu_{x+1} \ldots\ldots\ldots\ldots (5).$$
Of this a solution is
$$\nu_x = C\left(\frac{p}{q}\right)^{\frac{x}{2}} \sin(\mu x + \omega) \ldots\ldots\ldots\ldots (6),$$
where C and ω are arbitrary, and μ such that
$$2\sqrt{(pq)} \cos\mu = \alpha \ldots\ldots\ldots\ldots (7):$$
this form of solution is therefore real if α^2 is less than $4pq$. In order that (4) may satisfy the conditions (2), we must have
$$\nu_0 = C\sin\omega = 0, \quad \nu_a = C\left(\frac{p}{q}\right)^{\frac{a}{2}} \sin(\mu a + \omega) = 0 \ldots (8).$$
It is impossible to satisfy these two conditions without making $C = 0$, which would give a nugatory result, unless $\sin \mu a = 0$ or $\mu = \frac{r\pi}{a}$, r being an integer. Let us therefore assume this value for μ; and then, by (7),
$$\alpha = 2\sqrt{(pq)} \cos\frac{r\pi}{a} \ldots\ldots\ldots\ldots (9).$$
In order to satisfy (8), we have now only to make $\omega = 0$, and then, substituting the values of ν_x and α in (4), we get
$$y_{xz} = (4pq)^{\frac{z}{2}} \left(\frac{p}{q}\right)^{\frac{x}{2}} C \sin\frac{r\pi}{a} x \left(\cos\frac{r\pi}{a}\right)^z \ldots (10).$$
This value, in which C and r are arbitrary, satisfies (1) and (2), and in consequence of the linearity of these equations they will be satisfied by a sum of similar values, and we shall thus have a more general solution, viz.
$$y_{xz} = (4pq)^{\frac{z}{2}} \left(\frac{p}{q}\right)^{\frac{x}{2}} \Sigma C \sin\frac{r\pi}{a} x \left(\cos\frac{r\pi}{a}\right)^z \ldots (11).$$
If in this we put $z = 0$, we have
$$y_{x0} = \left(\frac{p}{q}\right)^{\frac{x}{2}} \Sigma C \sin\frac{r\pi}{a} x \ldots\ldots\ldots\ldots (12).$$
Now, by (3), this is to be equal to p for $x = 1$, and to 0 for the $a - 2$ values of x, $2 . 3 \ldots . a - 1$. There are thus $a - 1$ conditions for (12) to fulfil, and therefore we have, extending the summation Σ from $r = 1$ to $r = a - 1$, the following system of equations:

$$\left.\begin{aligned}\sqrt{(pq)} &= C_1 \sin\frac{\pi}{a} + \dots + C_{a-1}\sin\frac{a-1}{a}\pi\\ 0 &= C_1\sin\frac{2\pi}{a} + \dots + C_{a-1}\sin 2\,\frac{a-1}{a}\pi\\ 0 &= \dots\dots\dots\dots\\ 0 &= C_1\sin\frac{a-1}{a}\pi + \dots + C_{a-1}\sin\frac{(a-1)^2}{a}\pi\end{aligned}\right\}\dots\dots(13).$$

From these $a-1$ equations we have to determine the $a-1$ quantities $C_1 \dots C_{a-1}$. In order to do this, multiply the first equation by $\sin\frac{r}{a}\pi$, the second by $\sin 2\frac{r}{a}\pi$, and so on, (r being an integer less than a), and add. Then, as may be easily shown, the coefficient of every one of the quantities C, except C_r, will in the resulting sum be equal to zero, while that of C_r will be $\frac{a}{2}$. Consequently (13) is equivalent to the system of equations included in the general formula

$$C_r = \frac{2}{a}\sqrt{(pq)}\sin\frac{r}{a}\pi\dots\dots\dots\dots(14),$$

and consequently (11) becomes

$$y_{xz} = \frac{1}{a}(4pq)^{\frac{z+1}{2}}\left(\frac{p}{q}\right)^{\frac{x}{2}}\Sigma_1^{a-1}\sin\frac{r}{a}\pi\sin\frac{rx}{a}\pi\left(\cos\frac{r}{a}\pi\right)^z\dots(15),$$

which is the required probability.

We may deduce from this formula by indirect considerations, one or two analytical theorems. For it is obviously impossible that the party should terminate in M's favour in less than x games, as x is the number of counters he must win from N. Consequently

$$\Sigma_1^{a-1}\sin\frac{r}{a}\pi\sin\frac{rx}{a}\pi\left(\cos\frac{r}{a}\pi\right)^z = 0\dots\dots(16)$$

for all integer values of z less than $x-1$.

Again, M may win the party at the x^{th} game, if he win x games in succession, the probability of which is p^x. Hence, putting $z = x-1$, we have

$$p^x = \frac{1}{a}(4pq)^{\frac{x}{2}}\left(\frac{p}{q}\right)^{\frac{x}{2}}\Sigma_1^{a-1}\sin\frac{r}{a}\pi\sin\frac{rx}{a}\pi\left(\cos\frac{r}{a}\pi\right)^{x-1},$$

$$\text{or}\quad \Sigma_1^{a-1}\sin\frac{r}{a}\pi\sin\frac{rx}{a}\pi\left(\cos\frac{r}{a}\pi\right)^{x-1} = \frac{a}{2^x}\dots\dots(17).$$

These formulæ may undoubtedly be established by other

methods, but I have thought it worth while to point out this way of deducing them, from the analogy it bears to that in which many remarkable theorems are obtained by Poisson, in his *Théorie de la Chaleur,* namely by considering the nature of the quantities which his formulæ represent. This mode of establishing analytical theorems by considerations founded on the interpretation of our results, is one of the most curious features of the more recent methods of treating physical questions.

To (16) and (17) another theorem may be added, by the following consideration. M, if he win, must win the party either in x games or in x + an even number of games. For if he lose k games he must win back k games and x more or there must have been $x + 2k$ games in the party. Hence his chance is zero whenever $z + 1 = x + 2k + 1$, and therefore

$$\Sigma_1^{a-1} \sin \frac{r}{a} \pi \sin \frac{rx}{a} \pi \left(\cos \frac{r}{a} \pi \right)^{x+2k} = 0 \ldots . (18),$$

k being any positive integer whatever.

When a is infinite, the sums contained in the three last equations become definite integrals. Let

$$\frac{r}{a} \pi = \phi, \quad \text{then } \frac{\pi}{a} = d\phi, \quad \text{and } \frac{a-1}{a} \pi = \pi.$$

Consequently (16), (17), (18), become respectively

$$\int_0^\pi \sin \phi \sin x\phi (\cos \phi)^z d\phi = 0 \ldots\ldots . (19),$$

(z being integral and less than $x - 1$),

$$\int_0^\pi \sin \phi \sin x\phi (\cos \phi)^{x-1} d\phi = \frac{\pi}{2^x} \ldots\ldots (20),$$

$$\int_0^\pi \sin \phi \sin x\phi (\cos \phi)^{x+2k} d\phi = 0 \ \ldots\ldots (21).$$

If, instead of seeking the probability that M will win the party *at* the $(z + 1)^{\text{th}}$ game, we wished to find that of his winning it *after* z or more games shall have been played, we should only have to sum (15) for z from z to infinity. Calling this new probability u_{xz}, we should thus get

$$u_{xz} = \frac{1}{a} (4pq)^{\frac{z+1}{2}} \left(\frac{p}{q}\right)^{\frac{x}{2}} \Sigma_1^{a-1} \frac{\sin \frac{r}{a} \pi \sin \frac{rx}{a} \pi}{1 - 2\sqrt{(pq)} \cos \frac{r}{a} \pi} \left(\cos \frac{r}{a} \pi \right)^z \ldots (22).$$

If in (22) we put z equal to zero, we have then the probability of M's winning the party at the first, second, &c.

games, *i.e.* of his winning it at all. Writing simply u_x for u_{x0}, we shall thus get

$$u_x = \frac{2}{a}\sqrt{(pq)}\left(\frac{p}{q}\right)^{\frac{x}{2}} \Sigma_1^{a-1} \frac{\sin\frac{r}{a}\pi \sin\frac{rx}{a}\pi}{1 - 2\sqrt{(pq)}\cos\frac{r}{a}\pi} \quad \ldots\ldots(23).$$

Now of this probability we can obtain, as is well known, a much simpler expression. For it is easily seen that we shall have

$$u_x = pu_{x-1} + qu_{x+1} \ldots\ldots\ldots\ldots\ldots(24)$$

for every value of x, provided that, instead of considering u_x as the probability that M *will* win the party, we make it denote the probability that he either *has* won or *will* win it. As it is impossible that he can have already won it while x differs from zero, this alteration does not affect the value represented by u_x except for the case of $x = 0$. In this case the value of u_x, as expressed by (23), will be zero, as the party is at an end, M having already won it. But according to the proposed modification, the new value of u_0 will be unity, and therefore we have for the initial and final values of u_x,

$$u_0 = 1, \quad u_a = 0 \ldots\ldots\ldots\ldots\ldots(25).$$

The necessity of this modification arises from this, that otherwise the relation expressed by (24) would not be in all cases true. For when $x = 1$, we should have $u_1 = qu_2$, whereas the true value is of course $u_1 = p + qu_2$.

From (24) we have (introducing the relation $p + q = 1$)

$$u_x = \alpha + \beta\left(\frac{p}{q}\right)^x \ldots\ldots\ldots\ldots\ldots(26),$$

α and β being arbitrary constants: and thence, by (25), we get

$$1 = \alpha + \beta,$$

$$0 = \alpha + \beta\left(\frac{p}{q}\right)^a,$$

and consequently $$u_x = \frac{\left(\frac{p}{q}\right)^x - \left(\frac{p}{q}\right)^a}{1 - \left(\frac{p}{q}\right)^a} \ldots\ldots\ldots\ldots\ldots(27).$$

This expression is therefore, except for $x = 0$, equivalent to (23), into which however the relation already mentioned, viz. that $p + q = 1$ has not as yet been introduced.

When p and q are equal, (27) becomes

$$u_x = \frac{a-x}{a} \dots\dots\dots\dots\dots (28),$$

while (23) similarly becomes

$$u_x = \frac{1}{a}\Sigma_1^{a-1} \frac{\sin\frac{r}{a}\pi \sin\frac{rx}{a}\pi}{1-\cos\frac{r}{a}\pi},$$

$$\text{or} \quad u_x = \frac{1}{a}\Sigma_1^{a-1} \cot\frac{r}{a}\frac{\pi}{2}\sin\frac{rx}{a}\pi \dots\dots\dots (29).$$

Comparing (28) and (29), we have the following theorem: writing x for $a-x$,

$$x = \Sigma_1^{a-1} \pm \cot\frac{r}{a}\frac{\pi}{2}\sin\frac{rx}{a}\pi \dots\dots\dots (30),$$

the upper sign to be taken when r is odd.

This theorem, like the preceding ones (16), (17), &c., requires x not to transgress the limits $x = 1$, $x = a-1$. In the case supposed (viz. when p and q are each equal to $\frac{1}{2}$), (22) becomes

$$u_{xz} = \frac{1}{a}\Sigma_1^{a-1} \cot\frac{r}{a}\frac{\pi}{2}\sin\frac{rx}{a}\pi\left(\cos\frac{r}{a}\pi\right)^z \dots . (31).$$

But as the party cannot be won in less than x games, $u_{x0} = u_{xz}$ while z is less than x, and therefore

$$x = \Sigma_1^{a-1} \pm \cot\frac{r}{a}\frac{\pi}{2}\sin\frac{rx}{a}\pi\left(\cos\frac{r}{a}\pi\right)^z \dots . (32),$$

of which (30) is a particular case.

If, instead of seeking the probability that at the $(z+1)^{\text{th}}$ game N would lose the party, by losing the last of his x counters, we had sought that of his having at the termination of this game any assigned number of counters k, the following method might have been made use of.

Let y_{xz} be the probability in question. It is clear that it will satisfy, as before, equations (1) and (2). But instead of (3), we shall in this case have

$$y_{x\,0} = 0, \text{ unless } x = k \pm 1, \text{ and } y_{k\pm1\cdot0} = \tfrac{1}{2}\{p + q \pm (p-q)\} \dots (3).$$

Equation (11) therefore, which depends merely on (1) and (2), will still obtain; but instead of the system of equations (13), we shall have the following:

$$\left.\begin{aligned}
0 &= C_1 \sin\frac{\pi}{a} + \ldots + C_{a-1}\sin\frac{a-1}{a}\pi \\
\&c. &= \&c. \\
\left(\frac{q^{k+1}}{p^{k-1}}\right)^{\frac{1}{2}} &= C_1 \sin\frac{k-1}{a}\pi + \ldots + C_{a-1}\sin\frac{(k-1)(a-1)}{a}\pi \\
0 &= \&c. \\
\left(\frac{q^{k+1}}{p^{k-1}}\right)^{\frac{1}{2}} &= C_1 \sin\frac{k+1}{a}\pi + \ldots + C_{a-1}\sin\frac{(k+1)(a-1)}{a}\pi \\
\&c. &= \&c. \\
0 &= C_1 \sin\frac{a-1}{a}\pi + \ldots + C_{a-1}\sin\frac{(a-1)^2}{a}\pi
\end{aligned}\right\} \ldots (13').$$

From whence, by the same system of factors as before, we deduce the general formula

$$C_r = \frac{4}{a}\left(\frac{q^{k+1}}{p^{k-1}}\right)^{\frac{1}{2}} \sin\frac{kr}{a}\pi \cos\frac{r}{a}\pi \ldots\ldots (14');$$

for the factors corresponding to the two equations whose first members are different from zero, are $\sin\frac{(k-1)r}{a}\pi$ and $\sin\frac{(k+1)r}{a}\pi$, and the sum of these is $2\sin\frac{kr}{a}\pi\cos\frac{r}{a}\pi$. Consequently the expression of the probability sought will be (accenting the y for distinctness),

$$y'_{xz} = \frac{2}{a}(4pq)^{\frac{z+1}{2}}\left(\frac{p}{q}\right)^{\frac{x-k}{2}} \Sigma_1^{a-1} \sin\frac{kr}{a}\pi \sin\frac{rx}{a}\pi \left(\cos\frac{r}{a}\pi\right)^{z+1} \ldots (15').$$

It is an obvious consequence of the discontinuity of the limiting conditions of the problem, that this expression does not reduce itself to (15) when k is taken equal to zero. For the same reason it is not applicable when k is equal to unity: and on the other hand, it is not to be greater than $a - 2$.

It is unnecessary to trace the different corollaries deducible from the last written equation, as it has been introduced merely to illustrate the facility with which our method discusses any proposed modification of the question of the duration of play.

One point, which is perhaps worth notice, is the symmetrical manner in which x and p, k and q, enter into (15'): the result, however, which is the interpretation of this symmetry may probably be obtained by general considerations.

A more general question would arise from supposing it possible for M to win or lose at each game any number of counters not greater than a. The method we have been

illustrating would apply to this question, but the solution of it involves that of an algebraical equation of a degree superior to the second.

Another part of the subject, namely, the numerical calculation of the expressions already obtained, would not be consistent with the design of this paper. When z is sufficiently large, all the summations with respect to r may be reduced to their first and last terms, unless a is extremely large, in which case other methods of approximating (those, namely, of Laplace), may be made use of.

Enough has probably been said to show the facility which the method I have proposed is capable of giving to questions of acknowledged difficulty. I am not aware that it has been before pointed out; but as I am not at present able to refer to any work on the subject, I cannot speak confidently on this point. [x, a are integral throughout.]

XII.—NOTE ON GEOMETRICAL DISCONTINUITY.

If parallel straight lines be drawn cutting an ellipse, and from the points of section normals be drawn, the intersections of the pairs of normals will all lie on an hyperbola concentric with the ellipse.

This problem presents a somewhat singular instance of geometrical discontinuity: the equation to the hyperbola will be found to be

$$(a^2x \sin \alpha - b^2y \cos \alpha)(x \cos \alpha - y \sin \alpha)$$
$$= \sin \alpha \cos \alpha\, (a^2 - b^2)^2 \left(\frac{a^2 \sin^2 \alpha - b^2 \cos^2 \alpha}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}\right)^2 \ldots (A),$$

where α is the angle at which the straight lines cut the axis major.

Now the above equation is found by considering the problem as that of finding the locus of the intersections of pairs of lines drawn according to an assigned law, and therefore we might say that the equation represented the locus of the intersections of the normals: this, however, would not be strictly correct, for the locus of the intersections is not an hyperbola, but only a small arc of an hyperbola, as may be seen without much difficulty. If the lines be drawn parallel to either of the axes, the hyperbola degenerates into two straight lines, but the intersections of the normals only occupy a finite portion of these lines, those portions, in fact, which lie between the centres of curvature at the extremities

of the axes, or between the cusps of the evolute; and more generally the portion of the hyperbola whose equation is (A), belonging to the problem, is the arc lying within the evolute of the ellipse. In the case of a circle $a = b$, and the hyperbola again degenerates into two straight lines, but the only portion of them belonging to the problem is their point of intersection.

H. G.

XIII.—NOTE ON THE LAW OF GRAVITY AT THE SURFACE OF A REVOLVING HOMOGENEOUS FLUID.

IT has been shown by Maclaurin that a homogeneous fluid, revolving uniformly round a fixed axis, and acted upon only by the attractive force of its own particles, may, with the same angular velocity, have two different figures of equilibrium, each a spheroid of revolution round the shorter axis: and Jacobi has shown that it may be in equilibrium in the form of an ellipsoid with three unequal axes, the shortest coinciding with the axis of rotation. The following simple consideration determines the law of gravity at the surface in each case.

Let any surface concentric with the free surface of the fluid, and similar to it, be described in the interior of the fluid. If all the fluid exterior to the surface were removed, the fluid would still be in equilibrium, since the *proportions* of the free surface depend only on the density and angular velocity. Hence the accelerating force at this surface, as far as it is due to the centrifugal force, and the attraction of the interior mass, must be everywhere perpendicular to the surface. But the mass without it, being contained between two concentric similar ellipsoids exerts no attraction on any point in the surface, and therefore the direction of the accelerating force on any point of this surface in the interior of the fluid is in the normal. Hence the surface must be of equal pressure. Now, let it be supposed to approach the free surface so as to be indefinitely near it. In order that the pressure on every point of it may be the same, the accelerating force on any point of the indefinitely thin shell between it and the free surface, or the force of gravity at any point of the free surface, must be inversely proportional to the thickness of the shell at the point, or inversely as the perpendicular from the centre to the tangent plane at the point. A very simple analytical proof of this result was given by Liouville (*Journal de Mathématiques*, vol. VIII. p. 360). We may also state it, that the force of gravity at any point of the free surface is inversely proportional to

the electrical tension at the point, supposing the surface an electrified conductor. P. Q. R.

XIV.—MATHEMATICAL NOTE.

Various solutions have appeared from time to time of the following problem: it is hoped that the one here given may claim attention from its simplicity.

"The circle which passes through the intersections of three tangents to a parabola, passes also through the focus."

Let
$$y = x\tan\theta_1 + m\cot\theta_1 \;\ldots\ldots\ldots\; (1),$$
$$y = x\tan\theta_2 + m\cot\theta_2 \;\ldots\ldots\ldots\; (2),$$
$$y = x\tan\theta_3 + m\cot\theta_3 \;\ldots\ldots\ldots\; (3),$$
be the equations to three tangents to a parabola making angles $\theta_1\ \theta_2\ \theta_3$ with the axis of x. Let x', y', be the intersection of (1) and (2); then we easily find

$$x' = \frac{m\cos\theta_1\cos\theta_2}{\sin\theta_1\sin\theta_2}, \qquad x' - m = m\frac{\cos(\theta_1+\theta_2)}{\sin\theta_1\sin\theta_2},$$
$$y' = m\frac{\sin(\theta_1+\theta_2)}{\sin\theta_1\sin\theta_2}.$$

Therefore

$$(x'-m)^2 + y'^2 = m^2\left\{\frac{\cos^2(\theta_1+\theta_2)}{\sin^2\theta_1\sin^2\theta_2} + \frac{\sin^2(\theta_1+\theta_2)}{\sin^2\theta_1\sin^2\theta_2}\right\}$$
$$= \frac{m}{\sin\theta_1\sin\theta_2}\cdot\frac{m\sin\theta_3}{\sin\theta_1\sin\theta_2\sin\theta_3}$$
$$= \frac{m}{\sin\theta_1\sin\theta_2}\cdot\frac{m\sin\{(\theta_1+\theta_2+\theta_3)-(\theta_1+\theta_2)\}}{\sin\theta_1 . \sin\theta_2\sin\theta_3}$$
$$= \frac{m\sin(\theta_1+\theta_2+\theta_3)}{\sin\theta_1\sin\theta_2\sin\theta_3}\cdot\frac{m\cos(\theta_1+\theta_2)}{\sin\theta_1\sin\theta_2}$$
$$- \frac{m\cos(\theta_1+\theta_2+\theta_3)}{\sin\theta_1\sin\theta_2\sin\theta_3}\cdot\frac{m\sin(\theta_1+\theta_2)}{\sin\theta_1\sin\theta_2}$$
$$= \frac{m\sin(\theta_1+\theta_2+\theta_3)}{\sin\theta_1\sin\theta_2\sin\theta_3}\cdot(x'-m) - \frac{m\cos(\theta_1+\theta_2+\theta_3)}{\sin\theta_1\sin\theta_2\sin\theta_3}\cdot y' \ldots(4).$$

As this relation is symmetrical with regard to $\theta_1\ \theta_2\ \theta_3$, it will hold at the other two intersections. Hence the circle of which the equation is (4) passes through the intersections of the three tangents of which the equations are (1), (2), (3); but (4) is evidently satisfied by $x = m$, $y = 0$, and therefore the circle passes through the focus.

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I.—ON THE THEORY OF LINEAR TRANSFORMATIONS.

By A. CAYLEY, Fellow of Trinity College.

THE following investigations were suggested to me by a very elegant paper on the same subject, published in the *Journal* by Mr. Boole. The following remarkable theorem is there arrived at. If a rational homogeneous function U, of the n^{th} order, with the m variables $x, y\ldots$, be transformed by linear substitutions into a function V of the new variables, $\xi, \eta\ldots$; if, moreover, θU expresses the function of the coefficients of U, which, equated to zero, is the result of the elimination of the variables from the series of equations $d_x U=0$, $d_y U=0$, &c., and of course θV the analogous function of the coefficients of V: then $\theta V = E^{n\alpha}.\theta U$, where E is the determinant formed by the coefficients of the equations which connect $x, y\ldots$ with $\xi, \eta\ldots$, and $\alpha=(n-1)^{m-1}$.* In attempting to demonstrate this very beautiful property, it occurred to me that it might be generalised by considering for the function U, not a homogeneous function of the n^{th} order between m variables, but one of the same order, containing n sets of (m) variables, and the variables of each set entering linearly. The form which Mr. Boole's theorem thus assumes is $\theta V = E_1^{\alpha}.E_2^{\alpha}\ldots E_n^{\alpha}.\theta U$. This it was easy to demonstrate would be true, if θU satisfied a certain system of partial differential equations. I imagined at first that these would determine the function θU, (supposed, in analogy with Mr. Boole's function, to represent the result of the elimination of the variables from $d_{x_1}U=0$, $d_{y_1}U=0\ldots$ $d_{x_2}U=0$, &c.): this I afterwards found was not the case; and thus I was led to a class of functions, including as a

* The value of α was left undetermined, but Mr. Boole has since informed me, he was acquainted with it at the time his paper was written; and has given it in a subsequent paper.

particular case the function θU, all of them possessed of the same characteristic property. The system of partial differential equations were without difficulty replaced by a more fundamental system of equations, upon which, assumed as definitions, the theory appears to me naturally to depend; and it is this view of it which I intend partially to develope in the present paper.

I have already employed the notation

$$\left\|\begin{matrix} \alpha, & \beta, & \gamma, & \delta \ldots \\ \alpha', & \beta', & \gamma', & \delta' \ldots \\ \alpha'', & \beta'', & \gamma'', & \delta'' \ldots \\ \vdots \end{matrix}\right\| \ldots\ldots\ldots (1)$$

(where the number of horizontal rows is less than that of vertical ones) to denote the series of determinants,

$$\left|\begin{matrix} \alpha, & \beta, & \gamma \ldots \\ \alpha', & \beta', & \gamma' \ldots \\ \alpha'', & \beta'', & \gamma'' \ldots \\ \vdots \end{matrix}\right| \ldots\ldots\ldots\ldots (2),$$

which can be formed out of the above quantities by selecting any system of vertical rows; these different determinants not being connected together by the sign +, or in any other manner, but being looked upon as perfectly separate.

The fundamental theorem for the multiplication of determinants gives, applied to these, the formula

$$\left\|\begin{matrix} A, & B, & C, & D \ldots \\ A', & B', & C', & D' \ldots \\ A'', & B'', & C'', & D'' \ldots \end{matrix}\right\| = E \left\|\begin{matrix} \alpha, & \beta, & \gamma, & \delta \ldots \\ \alpha', & \beta', & \gamma', & \delta' \ldots \\ \alpha'', & \beta'', & \gamma'', & \delta'' \ldots \end{matrix}\right\| \ldots (3),$$

where
$$\left.\begin{aligned} A &= \lambda\alpha + \lambda'\alpha' + \lambda''\alpha'' + \ldots \\ B &= \lambda\beta + \lambda'\beta' + \lambda''\beta'' + \ldots \\ &\vdots \\ A' &= \mu\alpha + \mu'\alpha' + \mu''\alpha'' + \ldots \\ B' &= \mu\beta + \mu'\beta' + \mu''\beta'' + \ldots \\ &\vdots \\ &\text{\&c.} \end{aligned}\right\} \ldots\ldots (4),$$

$$E = \left|\begin{matrix} \lambda, & \mu \ldots \\ \lambda', & \mu' \ldots \\ \vdots \end{matrix}\right| \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

And the meaning of the equation is, that the terms on the first side are equal, each to each, to the terms on the second side.

This preliminary theorem being explained, consider a set of arbitrary coefficients, represented by the general formula

$$rst\ldots \ldots\ldots\ldots\ldots (6),$$

in which the number of symbolical letters r, $s\ldots$ is n, and where each of these is supposed to assume all integer values, from 1 to m inclusively.

Let $$\alpha s_{,}'t_{,}'\ldots,\quad \alpha s_{,}''t_{,}''\ldots,\ \ldots \ldots\ldots (7)$$

represent the whole series, taken in any order, in which the first symbolical letter is α. Similarly,

$$r_{,,}'\alpha t_{,,}'\ldots,\quad r_{,,}''\alpha t_{,,}'',\ \ldots \ldots\ldots (8),$$

the whole series of those in which the second symbolical letter is α, and so on.

Imagine a function u of the coefficients, which is simultaneously of the forms

$$u = H_p \left\| \begin{matrix} 1s_{,}'t_{,}'\ldots, & 1s_{,}''t_{,}''\ldots, & \ldots \\ 2s_{,}'t_{,}'\ldots, & 2s_{,}''t_{,}''\ldots, & \ldots \\ \vdots & & \end{matrix} \right\| \ldots\ldots (\mathrm{A}),$$

$$u = H_p \left\| \begin{matrix} r_{,,}'1t_{,,}'\ldots, & r_{,,}''1t_{,,}''\ldots, & \ldots \\ r_{,,}'2t_{,,}'\ldots, & r_{,,}''2t_{,,}''\ldots, & \ldots \\ \vdots & & \end{matrix} \right\|$$

&c.; in which H_p denotes a rational homogeneous function of the order p. The function H is not necessarily supposed to be the same in the above equations, and in point of fact it will not in general be so. The number of equations is of course (n).

The function (u), whose properties we proceed to investigate, may conveniently be named a "Hyperdeterminant." Any function satisfying any of the equations (A), without satisfying all of them, will be an "Incomplete Hyperdeterminant." But, considering in the first place such as are complete—

Let $\dot{r}st\ldots$ be a new set of coefficients connected with the former ones by a system of equations of the form

$$\dot{r}st\ldots = \lambda_1{}^r 1st\ldots + \lambda_2{}^r 2st\ldots + \lambda_m{}^r mst\ldots \ldots .(9),$$

(where the r in $\lambda_1{}^r\ldots$ is not an exponent, but an affix).

Suppose $\dot{u}$ is the same function of these new coefficients

that u was of the former ones. Then consider the first of the equations (A) and the equation (3), and writing

$$L = \begin{vmatrix} \lambda_1^{\,1}, & \lambda_1^{\,2} \dots \\ \lambda_2^{\,1}, & \lambda_2^{\,2} \dots \\ \vdots & \end{vmatrix} \dots\dots\dots\dots (10),$$

we have immediately the equation

$$\dot{u} = L^p . u. \dots\dots\dots\dots\dots (11).$$

Consider the new set of coefficients

$$\ddot{r}\dot{s}t\dots = \mu_1^{\,s} \dot{r}1t\dots + \mu_2^{\,s} \dot{r}2t\dots + \dots \mu_m^{\,s} \dot{r}mt\dots \dots\dots (12),$$

and $\ddot{u}$ the analogous function of these; then, from the second of the equations (A) and the equation (3), and writing

$$M = \begin{vmatrix} \mu_1^{\,1}, & \mu_1^{\,2} \dots \\ \mu_2^{\,1}, & \mu_2^{\,2} \dots \\ \vdots & \end{vmatrix} \dots\dots\dots\dots (13),$$

$$\ddot{u} = M^p . \dot{u} = L^p M^p u. \dots\dots\dots\dots (14).$$

In like manner, considering the new coefficients $\dddot{r}\dot{s}\dot{t}\dots$, where

$$\dddot{r}\dot{s}\dot{t}\dots = \nu_1^{\,t} \ddot{r}\dot{s}1\dots + \nu_2^{\,t} \ddot{r}\dot{s}2\dots \dots + \nu_n^{\,t} \ddot{r}\dot{s}m\dots \dots\dots (15),$$

the new function $\dddot{u}$ and the quantity N, given by

$$N = \begin{vmatrix} \nu_1^{\,1}, & \nu_1^{\,2} \dots \\ \nu_2^{\,1}, & \nu_2^{\,2} \dots \\ \vdots & \end{vmatrix} \dots \quad \dots\dots (16),$$

we have, as before,

$$\dddot{u} = N^p \dddot{u} = L^p M^p N^p u \dots\dots\dots (17),$$

or

$$\dddot{u} = L^p M^p N^p u. \dots\dots\dots\dots\dots (18);$$

whence, generally, denoting the last result by u',

$$u' = L^p M^p N_p \dots u \dots\dots\dots\dots\dots (B), (1).$$

Consider now the function

$$\Sigma\Sigma\Sigma\dots (rst\dots x_r y_s z_t \dots). \dots\dots\dots (19),$$

where the Σ's refer successively to $r, s, t\dots$, and denote summations from 1 to m inclusively. If u be looked upon as a derivative from the above function, we may write

$$u = \Theta . \Sigma\Sigma\Sigma\dots (rst\dots x_r y_s z_t \dots). \dots\dots (20).$$

Assume

$$\left.\begin{array}{l} x_r = \lambda_r^{\,1} \dot{x}_1 + \lambda_r^{\,2} \dot{x}_2 \dots + \lambda_r^{\,m} \dot{x}_m \\ y_s = \mu_s^{\,1} \dot{y}_1 + \mu_s^{\,2} \dot{y}_2 \dots + \mu_s^{\,m} \dot{y}_m \\ \vdots \end{array}\right\} \dots\dots (21).$$

It is easy to obtain

$$\Sigma\Sigma\Sigma\ldots(rst\ldots x_r y_s z_t\ldots) = \Sigma\Sigma\Sigma\ldots(\dot{r}st\ldots \dot{x}_r y_s z_t\ldots)\ldots$$
$$= \Sigma\Sigma\Sigma\ldots(\dot{r}\dot{s}\dot{t}\ldots \dot{x}_r \dot{y}_s \dot{z}_t\ldots)\ldots \ldots\ldots(22),$$

and the formula for (u) becomes

$$\Theta\,\Sigma\Sigma\Sigma\ldots(\dot{r}\dot{s}\dot{t}\ldots \dot{x}_r \dot{y}_s \dot{z}_t)\ldots = L^p M^p N^p \ldots \Theta\,\Sigma\Sigma\Sigma\ldots(rst\ldots x_r y_s z_t\ldots)$$
$$\ldots\ldots(\mathrm{B}),\ (2).$$

Proceeding to obtain the expression of the coefficients $\dot{r}\dot{s}\dot{t}\ldots$ in terms of the coefficients $rst\ldots$, we have

$$\dot{r}\dot{s}\dot{t}\ldots = \Sigma\Sigma\Sigma\ldots(\lambda_f^r \mu_g^s \nu_h^t \ldots fgh\ldots)\ldots\ldots(\mathrm{C}),$$

where the Σ's refer successively to f, g, $h\ldots$, denoting summations from 1 to m inclusively. Having this equation, it is perhaps as well to retain

$$u' = L^p M^p N^p \ldots u \ldots\ldots\ldots\ldots(\mathrm{B}),\ (1),$$

instead of (B, 2), that form being principally useful in showing the relation of the function (u) to the theory of the transformation of functions.

It may immediately be seen, that in the equations (B, C) we may, if we please, omit any number of the marks of variation (.), omitting at the same time the corresponding signs Σ, and the corresponding factors of the series L, M, $N\ldots$

Also, if u be such as only to satisfy some of the equations (A); then, if in the same formulæ we omit the corresponding marks (.), summatory signs, and terms of the series L, M, $N\ldots$, the resulting equations are still true.

From the formulæ (A) we may obtain the partial differential equations

$$\Sigma\Sigma\ldots\left(\alpha st\ldots\frac{d}{d\beta st\ldots}\right)u = 0, \text{ or } pu\ldots\ldots(\mathrm{D}),$$
$$\Sigma\Sigma\ldots\left(r\alpha t\ldots\frac{d}{dr\beta t\ldots}\right)u = 0, \text{ or } pu,$$

according as α is not equal or is equal to β;

and so on: the summatory signs referring in every case to those of the series r, s, $t\ldots$, which are left variable, and extending from 1 to m inclusively.

To demonstrate this, consider the general form of u, as given by the first of the equations (A). This is evidently composed of a series of terms, each of the form

$$cPQR\ldots(p \text{ factors}).$$

In which $P = \begin{vmatrix} 1s_{,}'t_{,}', & 1s_{,}''t_{,}'' \; \ldots . \; (m \text{ terms}) \\ \vdots & \\ \alpha s_{,}'t_{,}', & \alpha s_{,}''t_{,}'' \\ \vdots & \\ \beta s_{,}'t_{,}', & \beta s_{,}''t_{,}'' \\ \vdots & \end{vmatrix}$

Q, R, &c. being of the same form,

$$\Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st \ldots}\right) u$$

$$= cQR \ldots \Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st \ldots}\right) P + \&c. + \&c.,$$

and

$$\Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st \ldots}\right) P = \begin{vmatrix} 1s_{,}'t_{,}' \ldots 1, & s_{,}''t_{,}'', \ldots (m \text{ terms}) \\ \vdots & \\ \alpha s_{,}'t_{,}' \ldots, & \alpha s_{,}''t_{,}'', \ldots \\ \vdots & \\ \alpha s_{,}'t_{,}' \ldots, & \alpha s_{,}''t_{,}'', \ldots \\ \vdots & \end{vmatrix} = 0;$$

so that all the terms on the second side of the equation vanish. If, however, $\beta = \alpha$,

$$\Sigma\Sigma \ldots \left(\alpha st \ldots \frac{d}{d\beta st \ldots}\right) P = P;$$

whence, on the second side, we have

$$cQR \ldots P + cPR \ldots Q + \&c. = p \,.\, cPQR \ldots + \&c. + \&c.$$
$$= pU,$$

or the theorem in question is proved.

In the case of an incomplete hyperdeterminant, the corresponding systems of equations are of course to be omitted. In every case it is from these equations that the form of the function (u) is to be investigated; they entirely replace the system (A).

A very important case of the general theory is, when we suppose the coefficients $rst \ldots$ to have the property $r's't' \ldots = r''s''t'' \ldots$, whenever $r's't' \ldots$ and $r''s''t'' \ldots$ denote the same combination of letters; and also that the coefficients λ are equal to the coefficients μ, $\nu \ldots$, each to each. In this case the coefficients $\dot{r}\dot{s}\dot{t} \ldots$ have likewise the same property, viz. that $\dot{r}''\dot{s}''\dot{t}'' \ldots = \dot{r}'\dot{s}'\dot{t}' \ldots$, whenever $r's't' \ldots$ and $r''s''t'' \ldots$ denote the same combination of letters.

The equations (B, 1), (B, 2), become in this case

$$u' = L^{np} u \; \ldots \ldots \ldots \ldots \ldots \; (\text{B}, 3),$$

$$\Theta\,\Sigma\Sigma\Sigma\ldots\left\{\frac{[n]^n}{[\alpha]^\alpha.[\beta]^\beta\ldots}\dot{r}\dot{s}\dot{t}\ldots\dot{x}_r\dot{x}_s\dot{x}_t\ldots\right\}$$
$$=L^{np}.\,\Theta\left\{\frac{[n]^n}{[\alpha]^\alpha.[\beta]^\beta\ldots}rst\ldots x_r x_s x_t\ldots\right\}\ \ldots.\ (\mathrm{B},\,4).$$

Where only different combinations of values are to be taken for $r, s, t \ldots$ and $\alpha, \beta \ldots$, express how often the same number occurs in the series. In the equation (C), μ, ν must be replaced by λ, the equations (D) are no longer satisfied, the equations (A) reduce themselves to a single one, (so that there can be no question here of incomplete hyperdeterminants): but this is no longer sufficient to determine the function sought after. For this reason, the particular case, treated separately, would be far more difficult than the general one; but the formulæ of the general case being first established, these apply immediately to the particular one.* The case in question may be defined as that of symmetrical hyperdeterminants (a denomination already adopted for ordinary determinants). It would be easily seen what on the same principle would be meant by partially symmetrical hyperdeterminants.

I have not yet succeeded in obtaining the general expression of a hyperdeterminant; the only cases in which I can do so are the following: I. $p=1$, n even, (if n be odd, there only exist incomplete hyperdeterminants). II. $p=2$, $m=2$, n even. III. $p=3$, $m=2$, $n=4$.

I. The first case is, in fact, that of the functions considered at the termination of a paper in the *Cambridge Philosophical Transactions*, Vol. VIII. Part I.; though at that time I was quite unacquainted with the general theory.

Using the notation there employed, we have

$$u=\left\{\begin{matrix}\dagger \\ 11\ldots(n) \\ 22 \\ :\,: \\ mm\end{matrix}\right\},$$

a complete hyperdeterminant when n is even; and when n is odd the functions

$$\left\{\begin{matrix}\dagger \\ 11\ldots(n) \\ 22 \\ :\,: \\ mm\end{matrix}\right\},\qquad\left\{\begin{matrix}\ \ \dagger \\ 11\ldots(n) \\ 22 \\ :\,: \\ mm\end{matrix}\right\}$$

are each of them incomplete hyperdeterminants.

* See Note at the end of this paper.

(A) In the case of $n=2$, the complete hyperdeterminant is simply the ordinary determinant

$$\begin{vmatrix} 11, & 12\ldots & 1m \\ 21, & 22\ldots & 2m \\ \vdots & & \\ m1, & m2\ldots & mm \end{vmatrix}.$$

Stating the general conclusion as applied to this case, which is a very well known one,

"If the function $\quad U = \Sigma\Sigma\,(rs\,.\,x_r y_s)$

be transformed into a similar function

$$\Sigma\Sigma\,(\dot{r}\dot{s}\,.\,\dot{x}_r\dot{y}_s),$$

by means of the substitutions

$$x_r = \lambda_r^1\dot{x}_1 + \lambda_r^2\dot{x}_2\ldots + \lambda_r^m.\,\dot{x}_m,$$
$$y_s = \mu_s^1\dot{y}_1 + \mu_s^2\dot{y}_2\ldots + \mu_s^m.\,\dot{y}_m;$$

then $\begin{vmatrix} \dot{1}\dot{1} & \dot{1}\dot{2}\ldots \\ \dot{2}\dot{1} & \dot{2}\dot{2} \\ \vdots & \end{vmatrix} = \begin{vmatrix} \lambda_1^1, & \lambda_2^1\ldots \\ \lambda_1^2, & \lambda_2^2\ldots \\ \vdots & \end{vmatrix} \begin{vmatrix} \mu_1^1, & \mu_2^1\ldots \\ \mu_1^2, & \mu_2^2\ldots \\ \vdots & \end{vmatrix} \begin{vmatrix} 11, & 12\ldots \\ 21, & 22\ldots \\ \vdots & \end{vmatrix}$

Also, by what has preceded,

$$\dot{r}\dot{s} = \Sigma\Sigma\,(\lambda_f^r\,.\,\mu_g^s\,.fg);$$

so that the theorem is easily seen to amount to the following one—"If the terms of a determinant of the m^{th} order be of the form $\Sigma_r\Sigma_s(rs\,.\,x_{r.\rho}y_{s.\sigma})$, r, s extending as before, from 1 to m inclusively, the determinant itself is the product of three determinants; the first formed with the coefficients rs, the second with the quantities x, and the third with the quantities y."

In a following number of the *Journal* I shall prove, and apply to the theories of Maxima and Minima and of Spherical Co-ordinates, (I may just mention having obtained, in an elegant form, the formulæ for transforming from one oblique set of co-ordinates to another oblique one) the more general theorem,

"If k be the order of the determinant formed as above, the determinant itself is a quadratic function, its coefficients being determinants formed with the coefficients rs, its variibles being determinants formed respectively with the variables x and the variables y; and the number of variables in each set being the number of combinations of k things out of m, ($=1$ if $k=m$, if $k>m$ the determinant vanishes)."

I shall give in the same paper the demonstration of a very beautiful theorem, rather relating, however, to determinants than to quadratic functions, proved by Hesse in a Memoir in *Crelle's Journal*, vol. XX., "De curvis et superficiebus secundi ordinis;" and from which he has deduced the most interesting geometrical results. Another Memoir, by the same author, *Crelle*, vol. XXVIII., "Ueber die Elimination der Variabeln aus drei algeb. Gleichungen vom zweiter Grade mit zwei Variabeln," though relating in point of fact rather to functions of the third order, contains some most important results. A few theorems on quadratic functions, belonging, however, to a different part of the subject, will be found in my paper already quoted in the *Cambridge Philosophical Transactions;* and likewise in a paper in the *Journal* on the Algebraical Geometry of (n) dimensions.

I shall, just before concluding this case, write down the particular formula corresponding to three variables, and for the symmetrical case. It is, as is well known, the theorem,

"If $\quad U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy$

be transformed into

$$\mathfrak{A}\xi^2 + \mathfrak{B}\eta^2 + \mathfrak{C}\theta^2 + 2\mathfrak{F}\eta\theta + 2\mathfrak{G}\xi\theta + 2\mathfrak{H}\xi\eta$$

by means of
$$x = \alpha\xi + \beta\eta + \gamma\theta,$$
$$y = \alpha'\xi + \beta'\eta + \gamma'\theta,$$
$$z = \alpha''\xi + \beta''\eta + \gamma''\theta.$$

Then $(\mathfrak{A}\mathfrak{B}\mathfrak{C} - \mathfrak{A}\mathfrak{F}^2 - \mathfrak{B}\mathfrak{G}^2 - \mathfrak{C}\mathfrak{H}^2 + 2\mathfrak{F}\mathfrak{G}\mathfrak{H})$

$$(\alpha\beta'\gamma'' - \alpha\beta''\gamma' + \alpha'\beta''\gamma - \alpha'\beta\gamma'' + \alpha''\beta\gamma' - \alpha''\beta'\gamma)^2$$
$$(ABC - AF^2 - BG^2 - CH^2 + 2FGH).$$

(B) Let $n = 3$, and for greater simplicity $m = 2$; write

$$a = 111, \qquad e = 112,$$
$$b = 211, \qquad f = 212,$$
$$c = 121, \qquad g = 122,$$
$$d = 221, \qquad h = 222.$$

so that
$$U = ax_1y_1z_1 + bx_2y_1z_1 + cx_1y_2z_1 + dx_2y_2z_1$$
$$+ ex_1y_1z_2 + fx_2y_1z_2 + gx_1y_2z_2 + hx_2y_2z_2.$$

There is no complete hyperdeterminant (i.e. for $p = 1$), and the incomplete ones are

$$ah - bg - cf + de = u_{,}, \text{ suppose,}$$
$$ah - de - bg + cf = u_{,,},$$
$$ah - cf - de + bg = u_{,,,},$$

Thus, suppose the transforming equations are

$$x_1 = \lambda_1^{\ 1}\dot{x}_1 + \lambda_1^{\ 2}\dot{x}_2,$$
$$x_2 = \lambda_2^{\ 1}\dot{x}_1 + \lambda_2^{\ 2}\dot{x}_2;$$
$$y_1 = \mu_1^{\ 1}\dot{y}_1 + \mu_1^{\ 2}\dot{y}_2,$$
$$y_2 = \mu_2^{\ 1}\dot{y}_1 + \mu_2^{\ 2}\dot{y}_2;$$
$$z_1 = \nu_1^{\ 1}\dot{z}_1 + \nu_1^{\ 2}\dot{z}_2,$$

Then
$$z_2 = \nu_2^{\ 1}\dot{z}_1 + \nu_2^{\ 2}\dot{z}_2.$$

$\dddot{u}_{\prime} = (\mu_1^{\ 1}\mu_2^{\ 2} - \mu_2^{\ 1}\mu_1^{\ 2})(\nu_1^{\ 1}\nu_2^{\ 2} - \nu_2^{\ 1}\nu_1^{\ 2})\,u_{\prime},$ where y, z are changed,

$\dddot{u}_{\prime\prime} = (\nu_1^{\ 1}\nu_2^{\ 2} - \nu_2^{\ 1}\nu_1^{\ 2})(\lambda_1^{\ 1}\lambda_2^{\ 2} - \lambda_2^{\ 1}\lambda_1^{\ 2})\,u_{\prime\prime},$ " z, x "

$\dddot{u}_{\prime\prime\prime} = (\lambda_1^{\ 1}\lambda_2^{\ 2} - \lambda_2^{\ 1}\lambda_1^{\ 2})(\mu_1^{\ 1}\mu_2^{\ 2} - \mu_2^{\ 1}\mu_1^{\ 2})\,u_{\prime\prime\prime},$ " x, y "

We might also have assumed

$$u_{\prime} = ad - bc, \qquad \text{or } eh - gf,$$
$$u_{\prime\prime} = af - be, \qquad \text{or } ch - dg,$$
$$u_{\prime\prime\prime} = ag - ce, \qquad \text{or } bh - df.$$

But these are ordinary determinants.

(*C*). $n = 4$, $m = 2$.

$$\begin{array}{ll} a = 1111, & i = 1112, \\ b = 2111, & j = 2112, \\ c = 1211, & k = 1212, \\ d = 2211, & l = 2212, \\ e = 1121, & m = 1122, \\ f = 2121, & n = 2122, \\ g = 2211, & o = 2212, \\ h = 2221, & p = 2222. \end{array}$$

$$\begin{aligned} U = {} & ax_1y_1z_1w_1 + bx_2y_1z_1w_1 + cx_1y_2z_1w_1 + dx_2y_2z_1w_1 \\ & + ex_1y_1z_2w_1 + fx_2y_1z_2w_1 + gx_2y_2z_1w_1 + hx_2y_2z_2w_1 \\ & + ix_1y_1z_1w_2 + jx_2y_1z_2w_2 + kx_1y_2z_1w_2 + lx_2y_2z_1w_2 \\ & + mx_1y_1z_2w_2 + nx_2y_1z_2w_2 + ox_2y_2z_1w_2 + px_2y_2z_2w_2, \end{aligned}$$

we have $\quad u = ap - bo - cn + dm - el + fk + gj - hi.$

So that, with the same sets of transforming equations as above, and the additional one,

$$w_1 = \rho_1^{\ 1}\dot{w}_1 + \rho_1^{\ 2}\dot{w}_2,$$
$$w_n = \rho_2^{\ 1}\dot{w}_1 + \rho_2^{\ 2}\dot{w}_2,$$

we have

$$\ddddot{u} = (\lambda_1^{\ 1}\lambda_2^{\ 2} - \lambda_2^{\ 1}\lambda_1^{\ 2})(\mu_1^{\ 1}\mu_2^{\ 2} - \mu_2^{\ 1}\mu_1^{\ 2})(\nu_1^{\ 1}\nu_2^{\ 2} - \nu_2^{\ 1}\nu_1^{\ 2})(\rho_1^{\ 1}\rho_2^{\ 2} - \rho_2^{\ 1}\rho_1^{\ 2}) \cdot u.$$

This is important when viewed in reference to a result which will presently be obtained.

If we take the symmetrical case, we have

$$U=\alpha x^4+4\beta x^3y+6\gamma x^2y^2+4\delta xy^3+\varepsilon y^4\,;$$

which is transformed into

$$U'=\alpha' x'^4+4\beta' x'^3y'+6\gamma'^2 x'^2y'^2+4\delta' x'y'^3+\varepsilon' y'^4\,;$$

by means of

$$x=\lambda x'+\mu y',$$
$$y=\lambda' x'+\mu' y'.$$

Then, if

$$u=\alpha\varepsilon-4\beta\delta+3\gamma^2,$$
$$u'=\alpha'\varepsilon'-4\beta'\delta'+3\gamma'^2,$$
$$u'=(\lambda\mu_{,}-\lambda_{,}\mu)^4\,.\,u.$$

II. Where $p=2$, $m=2$, n is odd.

The expression

$$u=\begin{vmatrix} \begin{Bmatrix} \overset{\dagger}{1}111\ldots(n) \\ 1222 \end{Bmatrix}, & \begin{Bmatrix} \overset{\dagger}{1}111\ldots(n) \\ 2222 \end{Bmatrix}, \\ \begin{Bmatrix} \overset{\dagger}{1}111\ldots(n) \\ 2222 \end{Bmatrix}, & \begin{Bmatrix} \overset{\dagger}{2}111\ldots(n) \\ 2222 \end{Bmatrix}, \end{vmatrix}$$

is a complete hyperdeterminant; and that over whichever row the mark (†) of nonpermutation is placed. The different expressions so obtained are not, however, all of them independent functions. Thus, in the following example, where $n=3$, the three functions are absolutely identical.

(A). $n=3$, notation as in I. (B).

$$u=a^2h^2+b^2g^2+c^2f^2+d^2e^2$$
$$-2ahbg-2ahcf-2ahde-2bgcf-2bgde-2cfde$$
$$+4adfg+4bech.$$

and then

$$\dddot{u}=(\lambda_1{}^1\lambda_2{}^2-\lambda_2{}^1\lambda_1{}^2)^2(\mu_1{}^1\mu_2{}^2-\mu_2{}^1\mu_1{}^2)^2(\nu_1{}^2\nu_2{}^2-\nu_2{}^1\nu_1{}^2)^2u.$$

This is in many respects an interesting example. We see that the function (u) may be expressed in the three following forms:

$$u=(ah-bg-cf+de)^2+4\,(ad-bc)(fg-eh)\ldots(1),$$
$$u=(ah-bg-de+cf)^2+4\,(af-be)\,(dg-ch)\ldots(2),$$
$$u=(ah-cf-de+bg)^2+4\,(ag-ce)\,(df-bh)\ldots(3),$$

which are indeed the direct results of the general form above given, the sign (†) being placed in succession over the different columns: and the three forms, as just remarked, are in this case identical.

We see from the first of these that u is of the second or third, from the second that u is of the first or third, from

the third that u is of the first or second of the three following forms:

$$u=H_2\left\|\begin{matrix} a, b, c, d \\ e, f, g, h\end{matrix}\right\|, \; u=H_2\left\|\begin{matrix} a, b, e, f \\ c, d, g, h\end{matrix}\right\|, \; u=H_2\left\|\begin{matrix} a, c, e, g \\ b, d, f, h\end{matrix}\right\|$$

which is as it should be.

The following is a singular property of u.

Let $\quad a' - \frac{1}{2}\frac{du}{da}, \quad b' = \frac{1}{2}\frac{du}{dv}, \; \ldots . \; h' = \frac{1}{2}\frac{du}{dh}.$

Then, u' being the same function of these new coefficients that u is of the former ones,

$$u' = u^3.$$

To prove this, write

$$p = ah - bg - cf + de, \quad q = (ad - bc), \quad r = eh - fg.$$

$$\begin{aligned} a_, &= ap - 2q \,.\, e, & e_, &= -\,2ra + pe, \\ b_, &= bp - 2q \cdot f, & f_, &= -\,2rb + pf, \\ c_, &= cp - 2q \cdot g, & g_, &= -\,2rc + pg, \\ d_, &= dp - 2q \,.\, h, & h_, &= -\,2rd + ph. \end{aligned}$$

We have, as a particular case of the general formula just obtained, $\quad u_, = (p^2 - 4qr)^2\, u = u^2 \,.\, u = u^3.$

Also
$$\begin{aligned} a_, &= \;\; h', & e_, &= \;\; d', \\ b_, &= -g', & f_, &= -c', \\ c_, &= -f', & g_, &= -\,b', \\ d_, &= \;\; e', & h_, &= \;\; a'; \end{aligned}$$

whence $u_, = u'$, i.e. $u' = u^3$.

There is no difficulty in showing also, that if $a'', b'' \ldots h''$ are derived from $a', b' \ldots h'$, as these are from $a, b \ldots h$, then

$$a'' = u^2 a, \; b'' = u^2 b, \ldots . h'' = u^2 h.$$

The particular case of this theorem, which corresponds to symmetrical values of the coefficients, is given by M. Eisenstein, *Crelle*, vol. XXVII., as a corollary to his researches on the cubic forms of numbers.

Considering this symmetrical case

$$U = \alpha x^3 + 3\beta x^2 y + 3\gamma x y^2 + \delta y^3,$$

$$u = \alpha^2\delta^2 - 6\alpha\delta\beta\gamma - 3\beta^2\gamma^2 + 4\beta^3\delta + 4\alpha^3\gamma.$$

So that if U be transformed into

$$U' = \alpha' x'^3 + 3\beta' x'^2 y' + 3\gamma' x' y'^2 + \delta' y'^3;$$

by means of
$$x = \lambda x' + \mu y',$$
$$y = \lambda_1 x' + \mu_1 y'.$$

And $\quad u' = \alpha'^2\delta'^2 - 6\alpha'\delta'\beta'\gamma' - 3\beta'^2\gamma'^2 + 4\beta'^3\delta' + 4\alpha'^3\gamma',$

$$u' = (\lambda\mu_1 - \lambda_1\mu)^6 . u.$$

III. $p = 3, \quad m = 2, \quad n = 4.$

Notation as in I (C),

$$\begin{aligned} n &= A\,(\mathfrak{A} + 3\mathfrak{B} + 3\mathfrak{C} + 6\mathfrak{D} + 6\mathfrak{E}) \\ &\quad - B(\mathfrak{C} + \mathfrak{D} - 3\mathfrak{E} - \mathfrak{F} + 2\mathfrak{G} + 3\mathfrak{H}), \end{aligned}$$

where A, B, are arbitrary constants, and $\mathfrak{A}$, $\mathfrak{B}$, &c.... $\mathfrak{H}$, are functions of the coefficients, given as follows:—

$$\mathfrak{A} = a^3p^3 - b^3o^3 - c^3n^3 + d^3m^3 - e^3l^3 + f^3k^3 + g^3j^3 - h^3i^3.$$

$$\begin{aligned} \mathfrak{B} = &- a^2p^2bo + b^2o^2ap + c^2n^2dm - d^2m^2cn + e^2l^2fk - elf^2k^2 - g^2j^2hi + gjh^2i^2 \\ &- a^2p^2cn + b^2o^2dm + c^2n^2ap - d^2m^2bo + e^2l^2gj - f^2k^2hi - g^2j^2el + h^2i^2fk \\ &- a^2p^2el + b^2o^2fk + c^2n^2gj - d^2m^2hi + e^2l^2ap - f^2k^2bo - g^2j^2cn + h^2i^2dm \\ &- a^2p^2hi + b^2o^2gj + c^2n^2fk - d^2m^2el + e^2l^2dm - f^2k^2cn - g^2j^2bo + h^2i^2ap. \end{aligned}$$

$$\begin{aligned} \mathfrak{C} = &+ a^2p^2dm - b^2o^2cn - c^2n^2bo + d^2m^2ap - e^2l^2hi + f^2k^2gj + fkg^2j^2 - h^2i^2el \\ &+ a^2p^2fk - b^2o^2el - c^2n^2hi + d^2m^2gj - e^2l^2bo + f^2k^2ap + g^2j^2dm - h^2i^2cn \\ &+ a^2p^2gj - b^2o^2hi - c^2n^2el + d^2m^2fk - e^2l^2cn + f^2k^2dm + g^2j^2ap - h^2i^2bo. \end{aligned}$$

$$\begin{aligned} \mathfrak{D} = &\; apbocn - apbodm - apcndm + bocndm - elfkgj + elfkhi + elgjhi - higjfk \\ &+ apboel - apbofk - cndmgj + dmcnhi - elfkap + elfkbo + gjhicn - higjdm \\ &- apbogj + apbohi + cndmel - dmcnkf + elfkcn - elfkdm - gjhiap + higjbo \\ &+ apcnel - bodmfk - cnapgj + dmbohi - elgjap + fkhibo + gjelcn - hifkdm \\ &- apcnfk + bodmel + cnaphi - dmboel + elgjbo - fkhiap - gjeldm + hifkcn \\ &- apdmel + bocnkf + cmbogj - dmaphi + elhiap - fkgjbo - gifkcn + hidmel. \end{aligned}$$

$$\mathfrak{E} = apdmfk - bocnel - bocnih + dmapgj - elhibo + fkapgj + gjfkdm - hielcn.$$

$$\begin{aligned} \mathfrak{F} = &\; a^2phjo - b^2goip - c^2nflm + d^2emkn - e^2dlkn + f^2ckml + g^2bjpi - h^2aijo \\ &- i^2phbg + j^2gah + k^2nfde - l^2mecf + m^2ldcf - n^2ckde\; o^2bjah + p^2aibg \\ &+ a^2phkn - b^2golm - c^2nfip + d^2emjo - e^2dljo + f^2ckip + g^2bjpl - h^2aink \\ &- i^2phcf + j^2ogde + k^2nfah - l^2mebg + m^2ldbg - n^2ckah - o^2bjde + p^2aicf \\ &+ a^2phlm - b^2gokn - c^2nfjo + d^2emip - e^2ldpi + f^2ckjo + g^2bjnk - h^2aiml \\ &- i^2phde + j^2gocf + k^2nfbg - l^2meah + m^2dlah - n^2ckbg - o^2bjcf + p^2aide \\ &+ a^2pdno - b^2cmpo - c^2bpmn + d^2aomn - e^2hjkl + f^2gilk + g^2flij - h^2ekji \\ &- i^2lfgh + j^2kehg + k^2jhef - l^2igfe + m^2pbcd - h^2aodc - o^2ndab + p^2mbca \\ &+ a^2plng - b^2hkmo - c^2ejpn + d^2fiom - e^2cpjl + f^2doik + g^2anlj - h^2bmki \\ &- i^2odfh + j^2pceg + k^2mhbf - l^2nage + m^2khbd\; n^2lgac - o^2ifbd + p^2jeca \\ &+ a^2plfo - b^2ekpo - c^2hjmn + d^2gimn - e^2bpkl + f^2aolk + g^2dnij - h^2cmji \\ &- i^2ndgh + j^2mchg + k^2pbef - l^2oafe + m^2jhcd - n^2igcd - o^2lfab + p^2keba. \end{aligned}$$

$$\begin{aligned}
\mathfrak{G} = {} & apbgkn-boalhm-cndiep+dmcjfo-elfocj+fkpied+gjhmla-hignbk \\
& -ihjocf+gjidep+kflamh-lekbng+mdnbkg-ncmhal-obpied+paofjc \\
& +apbglm-boahkn-cndejo+dmcfip-elcfip+fkedoj+gjhakn-hibgml \\
& -ihjode+gjipcf+kflmbg-leknah+mdknah-ncmlbg-obpicf+paojde \\
& +apcflg-bojehk-cnjehk+dmilgf-elmpbc+fkadno+gjodno-hipmbc \\
& -ihadno+gjbmcp+kfcbpm-eladno+mdjehk-ncilfg-obilfg+pahejk \\
& +apcflm-badkne-cnahjo+dmbgip-elbgip+fkahjo+gjednk-hicfml \\
& -ihknde+mdahjo+kfipbg-bocfml+gjcfml-ncpigb-leahjo+paknde \\
& +apidng-bojcmh-cnbkpe+dmaflo-elmhjc-fkidng+gjaflo-ihbkpe \\
& -ihaflo+gjbkpe+fkcjhm-elidng+mdbkpe-cnaflo-boidng+pahmcj \\
& +apidfo-bojcep-cnbkhm+dmalng-elmnbk-fkalng+gjidfo-ihpecj \\
& -ihalng+gjkbmh+fkcjpe-elidfo+mdjcep-ncidfo-obalng+pahmbk. \\
\mathfrak{H} = {} & a^2honl-b^2pgmk-c^2pfmj+d^2onie-e^2dpkj+f^2ilco+g^2blni-h^2amkj \\
& -i^2pdfg+j^2oech+k^2nbeh-l^2mafg+m^2blch-n^2kadg-o^2jadf+p^2ibec:
\end{aligned}$$

we have, as usual,

$$\ddddot{u}=(\lambda_1^{\ 1}\lambda_2^{\ 2}-\lambda_2^{\ 1}\lambda_1^{\ 2})^3.(\mu_1^{\ 1}\mu_2^{\ 2}-\mu_2^{\ 1}\mu_1^{\ 2})^3(\nu_1^{\ 1}\nu_2^{\ 2}-\nu^1_{\ 2}\nu_1^{\ 2})^3(\rho_1^{\ 1}\rho_2^{\ 2}-\rho_1^{\ 2}\rho_2^{\ 1})^3.u.$$

Particular forms of U are

$A=1,\ B=0$

$$u=\mathfrak{A}+3\mathfrak{B}+3\mathfrak{C}+6\mathfrak{D}+6\mathfrak{E}$$

$$=(ap-bo-cn+dm-el+fk+gj-hi)^3.=v^3. \text{ suppose}$$

$A=1,\ B=9.$

$$u=\mathfrak{A}+3\mathfrak{B}-6\mathfrak{C}-3\mathfrak{D}+33\mathfrak{E}+9\mathfrak{F}-18\mathfrak{G}-27\mathfrak{H}.$$

$$=\theta U \text{ suppose,}$$

where $\theta U=0$ is the result of the elimination of the variables from the equations $d_{x_1}U=0$, $d_{y_1}U=0$, $d_{z_1}U=0$, $d_{w_1}U=0$, $d_{x_2}U=0$, $d_{y_2}U=0$, $d_{z_2}U=0$, $d_{w_2}U=0$. In fact, by an investigation similar to Mr. Boole's, applied to a function such as U, it is shown that θU has the characteristic property of the function u: also in the present case (u) is the most general function of its kind, so that θU is obtained from U by properly determining the constant. This has been effected by comparing the value of u, in the symmetrical case, with the value of θU, in the same case, the expanded expression of which is given by Mr. Boole in the *Journal*, vol. IV. p. 169. Assuming $A=1$, the result was $B=9$.

The general form of u now becomes

$$u = \alpha\nu^3 + \beta\theta U,$$

in which α, β, are indeterminate.

We have $\quad \ddddot{u} = \alpha\ddddot{\nu^3} + \beta\theta\ddddot{U} = M(\alpha\nu^3 + \beta\theta U).$

$$M = (\lambda_1{}^1\lambda_2{}^2 - \lambda_2{}^1\lambda_1{}^2)^3.(\mu_1{}^1\mu_2{}^2 - \mu_2{}^1\mu_1{}^2)^3(\nu_1{}^1\nu_2{}^2 - \nu_2{}^1\nu_1{}^2)^3(\rho_1{}^1\rho_2{}^2 - \rho_2{}^1\rho_1{}^2)^3.$$

And thence $\qquad \ddddot{\nu^3} = M\nu^3,$

which coincides with a previous formula, and

$$\theta\ddddot{U} = M\theta U:$$

whence, eliminating M,

$$\frac{\theta\ddddot{U}}{\ddddot{\nu^3}} = \frac{\theta U}{\nu^3},$$

an equation which is remarkable as containing only the constants of U and $\ddddot{U}$: it is an equation of condition which must exist among the constants of $\ddddot{U}$ in order that this function may be derivable by linear substitutions from U.

In the symmetrical case, or where

$$U = \alpha x^4 + 4\beta x^3y + 6\gamma x^2y^2 + 4\delta xy^3 - \varepsilon y^4.$$

It has been already seen that ν is given by

$$\nu = \alpha\varepsilon - 4\beta\delta + 3\gamma^2.$$

Proceeding to form θU, we have

$\mathfrak{A} = \alpha^3\varepsilon^3 - 4\beta^3\delta^3 + 3\gamma^6.$

$\mathfrak{B} = 4(\alpha\varepsilon\beta^2\delta^2 - \alpha^2\varepsilon^2\beta\delta + 3\gamma^2\beta^2\delta^2 - 3\beta\delta\gamma^4).$

$\mathfrak{C} = 3(\alpha^2\varepsilon^2\gamma^2 + 2\gamma^6 + \alpha\varepsilon\gamma^4 - 4\beta^3\delta^3).$

$\mathfrak{D} = 6(\alpha\varepsilon\beta^2\delta^2 - 2\alpha\varepsilon\beta\delta\gamma^2 + 3\beta^2\gamma^2\delta^2 - 2\beta\delta\gamma^4).$

$\mathfrak{E} = 3\,\alpha\varepsilon\gamma^4 - 4\beta^3\delta^3 + \gamma^4.$

$\mathfrak{F} = 6(\alpha^2\delta^2\gamma\varepsilon + \varepsilon^2\beta^2\gamma\alpha - 2\beta^3\varepsilon\gamma\delta - 2\delta^3\alpha\beta\gamma - 4\beta^2\gamma^2\delta^2.$
$\qquad + 4.\beta\delta\gamma^4 + \gamma^3\beta^2\varepsilon + \gamma^3\alpha\delta^2).$

$\mathfrak{G} = 12(\alpha\varepsilon\beta\delta\gamma^2 - \alpha\beta\gamma\delta^3 - \varepsilon\gamma\delta\beta^3 + \gamma^3\alpha\delta^2 + \gamma^3\varepsilon\beta^2 + \beta\delta\gamma^4 - 2\beta^2\gamma^2\delta^2).$

$\mathfrak{H} = (\alpha^2\delta^4 + \varepsilon^2\beta^4 - 4\beta^2\varepsilon\gamma^3 - 4\alpha\gamma^3\delta^2 + 6\beta^2\gamma^2\delta^2).$

And these values give

$$\begin{aligned}\theta U = {} & \alpha^3\varepsilon^3 - 6\alpha\beta^2\delta^2\varepsilon - 12\alpha^2\beta\delta\varepsilon^2 - 18\alpha^2\gamma^2\varepsilon^2 - 27\alpha^2\delta^4 - 27\beta^4\varepsilon^2. \\ & + 36\beta^2\gamma^2\delta^2 + 54.\alpha^2\gamma\delta^2\varepsilon + 54.\alpha\beta^2\gamma\varepsilon^2 - 54\alpha\gamma^3\delta^2 - 54\beta^2\gamma^3\varepsilon \\ & - 64\beta^2\delta^3 + 81\alpha\gamma^4\varepsilon + 108\alpha\beta\gamma\delta^3 + 108\beta^3\gamma\delta\varepsilon - 180\alpha\beta\gamma^2\delta\varepsilon.\end{aligned}$$

So that this function, divided by $(\alpha\varepsilon - 4\beta\delta - 3\gamma^2)^3$, is invariable for all functions of the fourth order which can be deduced one from the other by linear substitutions. The function $\alpha\varepsilon - 4\beta\delta + 3\gamma^2$ occurs in other investigations: I have met with it in a problem relating to a homogeneous function of two variables, of any order whatever, α, β, γ, δ, ε signifying the fourth differential coefficients of the function. But this is only remotely connected with the present subject.

Since writing the above, Mr. Boole has pointed out to me that in the transformation of a function of the fourth order of the form $ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$—besides his function θu, and my quadratic function $ae - 4bd + 3c^2$,—there exists a function of the third order $ace - b^2e - ad^2 - c^3 + 2bdc$, possessing precisely the same characteristic property, and that, moreover, the function θu may be reduced to the form

$$(ae - 4bd + 3c^2)^3 - 27.(ace - ad^2 - eb^2 - c^3 + 2bdc)^2;$$

the latter part of which was verified by trial; the former he has demonstrated in a manner which, though very elegant, does not appear to be the most direct which the theorem admits of. In fact, it may be obtained by a method just hinted at by Mr. Boole, in his earliest paper on the subject, *Mathematical Journal*, vol. II. p. 70. The equations $d_x{}^2u = 0$, $d_xd_yu = 0$, $d_y{}^2u = 0$, imply the corresponding equations for the transformed function: from these equations we might obtain two relations between the coefficients, which, in the case of a function of the fourth order, are of the orders 3 and 4 respectively: these imply the corresponding relations between the coefficients of the transformed function. Let $A = 0$, $B = 0$, $A' = 0$, $B' = 0$, represent these equations; then, since $A = 0$, $B = 0$, imply $A' = 0$, we must have $A' = \Lambda A' + MB$, Λ, M, being functions of λ, λ', μ, &c. μ': but B being of the fourth order, while A, A' are only of the third order in the coefficients of u, it is evident that the term MB must disappear, or that the equation is of the form $A' = \Lambda A$. The function A is obviously the function which, equated to zero, would be the result of elimination of x^2, xy, y^2, considered as independent quantities from the equations $ax^2 + 2bxy + cy^2 = 0$, $bx^2 + 2cxy + dy^2 = 0$, $cx^2 + 2dxy + ey^2 = 0$, viz. the function given above. Hence the two functions on which the linear transformation of functions of the fourth order ultimately depend are the very simple ones

$$ae - 4bd + 3c^2,\ ace - ad^2 - eb^2 - c^3 + 2bdc,$$

the function of the sixth order being merely a derivative from these. The above method may easily be extended: thus, for instance, in the transformation of functions of any even order, I am in possession of several of the transforming functions; that of the fourth order, for functions of the sixth order, I have actually expanded: but it does not appear to contain the complete theory. Again, in the particular case of homogeneous functions of two variables, the transforming functions may be expressed as symmetrical functions of the roots of the equations $u = 0$, which gives rise to an entirely distinct theory. This, however, I have not as yet developed sufficiently for publication. There does not appear to be anything very directly analogous to the subject of this note, in my general theory: if this be so, it proves the absolute necessity of a distinct investigation for the present case, the one which I have denominated the symmetrical one.

II.—ON MAGIC SQUARES.

By R. Moon, M.A. Fellow of Queens' College.

The theory of Magic Squares has long exercised the ingenuity of mathematicians. It will be my object in the present paper, rather to unfold a simplification of the theory itself than to present any new or striking additions to it.

The ordinal numbers, from 1 to 25 inclusive, may be represented by the formula

$$1 + x + 5y,$$

where x and y are independent, and may respectively take any of the values 0. 1. 2. 3. 4. That such will be the case may be easily seen by arranging the numbers in the following manner:

y						
0	1	2	3	4	5	
1	6	7	8	9	10	
2	11	12	13	14	15	
3	16	17	18	19	20	
4	21	22	23	24	25	
	0	1	2	3	4	x

By means of the above formula we shall proceed to construct a magic square.

A	*B*	*C*
$1+x_0+5y_0$	$1+x_2+5y_3$	$1+x_4+5y_1$
$1+x_1+5y_1$	$1+x_3+5y_4$	$1+x_0+5y_2$
$1+x_2+5y_2$	$1+x_4+5y_0$	$1+x_1+5y_3$
$1+x_3+5y_3$	$1+x_0+5y_1$	$1+x_2+5y_4$
$1+x_4+5y_4$	$1+x_1+5y_2$	$1+x_3+5y_0$

D	*E*
$1+x_1+5y_4$	$1+x_3+5y_2$
$1+x_2+5y_0$	$1+x_4+5y_3$
$1+x_3+5y_1$	$1+x_0+5y_4$
$1+x_4+5y_2$	$1+x_1+5y_0$
$1+x_0+5y_3$	$1+x_2+5y_1$

The subscribed figures in the above columns denote the values to be assigned to x and y respectively. If the substitutions thus indicated be made, it will be found that the numbers contained in the columns A, B, C, D, E, (the columns being arranged side by side in the order of the letters), will constitute a magic square. This will be seen if we consider that,

(1) The sum of the numbers in each column

$$= 5 + (x_0 + x_1 + x_2 + x_3 + x_4) + 5\,(y_0 + y_1 + y_2 + y_3 + y_4).$$

(2) If we take the first number in each column and add them together, their sum will be equal to the sum of each column taken vertically; and so of the sums of the second, third, &c. numbers respectively.

(3) The sum of the first number of A, the second of B, the third of C, the fourth of D, and the fifth of E, will likewise be equal to the same quantity; as also will be the sum of the fifth of A, the fourth of B, the third of C, the second of D, and the first of E.

A slight inspection of the columns will shew how they may be successively derived one from another. It will also be seen that by the same rule that B is derived from A, C from B, and so on, we may also derive A from E; so that the arrangement is (if we may so term it) circular: and of course we can go backwards in the circle as well as forwards, *i.e.* we may derive B from C as easily as we can derive C from B.

It will also be seen that, the order of the columns being preserved, it is quite indifferent which we place at the side or which we begin with. It is further to be observed that the sole limitation to be attended to in the formation of the first or *generating* column is, that its sum must be

$$= 5 + (x_0 + x_1 + x_2 + x_3 + x_4) + 5\,(y_0 + y_1 + y_2 + y_3 + y_4).$$

The order in which the values of x and y are assigned is indifferent. It is only requisite that no two numbers of the generating column contain the same value of x or the same value of y.

The number of magic squares which may be formed according to the above method is equal to the number of ways in which the *generating* column may be formed, *i.e.* $=(5.4.3.2.1)^2$; or, if we consider that each square will substantially recur four times, the only difference being according as we take the numbers from right to left or from top to bottom, the number of different squares will be $= \left(\dfrac{5.4.3.2.1}{2}\right)^2$.

The following arrangement will likewise constitute a magic square:

A	B	C
$1 + x_3 + 5y_2$	$1 + x_4 + 5y_1$	$1 + x_0 + 5y_0$
$1 + x_4 + 5y_3$	$1 + x_0 + 5y_2$	$1 + x_1 + 5y_1$
$1 + x_0 + 5y_4$	$1 + x_1 + 5y_3$	$1 + x_2 + 5y_2$
$1 + x_1 + 5y_0$	$1 + x_2 + 5y_4$	$1 + x_3 + 5y_3$
$1 + x_2 + 5y_1$	$1 + x_3 + 5y_0$	$1 + x_4 + 5y_4$

D	E
$1 + x_1 + 5y_4$	$1 + x_2 + 5y_3$
$1 + x_2 + 5y_0$	$1 + x_3 + 5y_4$
$1 + x_3 + 5y_1$	$1 + x_4 + 5y_0$
$1 + x_4 + 5y_2$	$1 + x_0 + 5y_1$
$1 + x_5 + 5y_3$	$1 + x_1 + 5y_2$

The mode of successive formation in this case is obvious. The arrangement here, as in the former example, is circular; but we are not at liberty in this case, as in the last, to begin with any column in the series. C must necessarily be the middle column. The reason of this will readily appear. For if we take D for the middle column, the sum of the places in the diagonals of the square (which will be found by taking the sum of the first number of B, the second of C, the third of D, the fourth of E, and the fifth of A, and the sum of the

fifth of B, the fourth of C, the third of D, the second of E, and the first of A,) will be respectively

$$5 + (x_4 + x_1 + x_3 + x_0 + x_2) + 5\,(y_1 + y_1 + y_1 + y_1 + y_1),$$
$$5 + (x_3 + x_3 + x_3 + x_3 + x_3) + 5\,(y_0 + y_3 + y_1 + y_4 + y_2).$$

Hence this arrangement fails. But if C be the middle column, the sums of the diagonals are, respectively,

$$5 + (x_3 + x_0 + x_2 + x_4 + x_1) + 5\,(y_2 + y_2 + y_2 + y_2 + y_2).$$
$$5 + (x_2 + x_2 + x_2 + x_2 + x_2) + 5\,(y_1 + y_4 + y_2 + y_0 + y_3):$$

which are severally equal to the sum of any vertical column, since

$$x_0 + x_1 + x_2 + x_3 + x_4 = 5x_2,$$
$$y_0 + y_1 + y_2 + y_3 + y_4 = 5y_2.$$

Hence we see that not only must C be the middle column, but the middle number of C must be $= 1 + x_2 + 5y_2$. From this it is easy to deduce that the number of different squares to be obtained from this method $= \left(\frac{4.3.2.1.}{2}\right)^2$.

In a similar manner it may be shewn that the number of magic squares which can be formed of the numbers 1. 2. 3 ...9

$$= \left(\frac{2.1}{2}\right)^2 = 1.$$

The number which can be formed of the numbers 1. 2. 3 ...49 by both methods

$$= 2.\left(\frac{7.6\ldots 3.2.1}{2}\right)^2 + \left(\frac{6.5\ldots 2.1}{2}\right)^2.$$

Let us now examine a little more particularly the nature of the method adopted in the above cases, and for this purpose let us revert to our original example. It will be seen that in that instance the column of x's in B is formed from that in A by rejecting the two first x's and throwing them to the bottom, their order being unchanged. The column of x's in C is formed from that in B in like manner, and so on for the rest.

In the second example the column of x's in B is formed from that in A by rejecting the first x only, and placing it at the bottom. The column of y's in the first case is formed by removing two of the y's from the foot of the preceding column and placing them in their order at the head of the new one, and similarly in the second case.

If we try the effect of rejection the *three* first x's in the column, we shall obtain the same succession of columns as in the first case, but in the reverse order: and if we reject the

four first x's we shall have the same succession of columns as in the second example, likewise in the reverse order.

When the magic square is to contain 49 places, we may reject the first x in the generating column, and so obtain $\left(\frac{6.5\ldots3.2.1}{2}\right)^2$ different squares. We may next reject the two first x's in the generating column, and so get $\left(\frac{7.6\ldots2.1}{2}\right)^2$ additional squares; and lastly we may reject the three first x's, and so obtain $\left(\frac{7.6.5\ldots2.1}{2}\right)^2$ squares.

But when the magic square contains 81 places, though we can make use of the two first of the above methods, yet when we reject the three first x's from the top of the column, and proceed by that rule, the method totally fails. It will be found that this failure is owing to the fact of the number of x's rejected being a divisor of the number of places in the side of the square. In this case however, if we reject the *four* first x's the method will succeed with certain limitations, *i.e.* provided the sum of the 2nd, 5th, and 8th places of x in the middle or generating column is equal to 12, and the sum of the like places of y is equal to the same number. Hence the number of squares obtained by this last method will be

$$= 4.\left(\frac{3.2.1 + 6.5\ldots2.1}{2}\right)^2,$$

and the whole number of magic squares of 81 places

$$= \left(\frac{9.8\ldots2.1}{2}\right)^2 + \left(\frac{8.7\ldots2.1}{2}\right)^2 + 4\left(\frac{3.2.1 \times 6.5\ldots2.1}{2}\right)^2.$$

Generally if $2n+1$ be a prime number, the whole number of squares of $(2n+1)^2$ places which can be performed by the above methods

$$= (n-1)\left\{\frac{(2n+1)\ldots3.2.1}{2}\right\}^2 + \left(\frac{2n\ldots3.2.1}{2}\right)^2.$$

When the number of places in a side of the magic square is not a prime number, the rule for finding the number of squares to be obtained by the above methods is one of very great complexity; and, as the subject is of no practical importance, I shall content myself with merely indicating the method by which it is to be obtained. Let mr be the number of places in the side of a square where r is a prime number. If we reject nr of the x's from the top of the generating column, and the same number of y's from the

bottom, the method totally fails (n denoting any integer). If we reject *one* x from the top and one y from the bottom, the method fails partially, *i.e.* we shall obtain only

$$\left\{\frac{(nr-1).(nr-2)\ldots.2.1}{2}\right\}^2 \text{squares.}$$

If we reject $(1+nr)x$'s and a like number of y's from the top and bottom respectively, there will again be a partial failure. If p denote the number of ways in which the number mr^2 can be made up by taking the sum of m of the numbers $0.1.2\ldots.(mr-1)$, the number of squares to be deduced by rejecting $(1+nr)x$'s and y's respectively will be

$$=\left\{\frac{p\times(mr-m)(mr-m-1)\ldots.3.2.1}{2}\right\}.$$

If we reject a number of x's and y's respectively, greater than unity and not divisible by either nr or $nr-1$, the resulting number of squares will be

$$=\left(\frac{mr.(mr-1)\ldots.3.2.1}{2}\right)^2.$$

I propose to consider the case of magic squares of an even number of places, in a subsequent paper.

III.—ON THE THEORY OF DEVELOPMENTS. PART I.

BY GEORGE BOOLE.

IN a paper published in the *Philosophical Transactions* for the year 1844, Part II., I had occasion to investigate the expansion of the binomial $f(\pi+\rho)$, on the assumption that π and ρ are symbols operating on a certain subject u, and combining according to the law

$$\rho f(\pi)u = f\{\phi(\pi)\}\rho u \ldots\ldots\ldots (1).$$

The result possesses a theoretic interest, because it shews the general form of the development of which Taylor's is a particular case; while at the same time it is, I conceive, of fundamental importance in the theory of differential equations and of equations of finite differences. For this application I must, however, refer to the paper above mentioned. My design here is to notice certain other deductions from the theorem in question, and to shew that the method by which it was obtained is generally applicable.

It may be proper, by way of introduction, to state in what sense such an expression as $f(\pi)$ is to be understood, when π is not a symbol of quantity.

In the first place, if $f(\pi) = \pi^m$, it is evident, that by $\pi^m u$ we are to understand the result obtained by operating with π upon u, then with π upon the result, and so on till the operation denoted by the symbol π shall have been m times performed.

If $f(\pi)$ is of the forms to which Maclaurin's theorem is applicable, it is evident that we must, in interpreting $f(\pi)$, suppose the expansion to be effected. Thus

$$(\sin \pi)\, u = \left(\pi - \frac{\pi^3}{1.2.3} + \frac{\pi^5}{1.2.3.4.5} - \&c.\right) u.$$

The legitimacy of the expansion of $f(\pi)$ is apparently independent of the nature of the symbol π; for as π operates solely on u, it may be regarded as commutative with respect to the constants in $f(\pi)$. We shall at any rate, in what follows, regard π as a symbol of this nature.

When $f(\pi)$ is not of the forms to which Maclaurin's expansion applies, it does not appear to be generally possible to interpret $f(\pi)\, u$. The laws of combination and of interpretation (when possible) to which $f(\pi)$ must be considered subject, have perhaps no other foundation than analogy.

In considering a binomial $f(\pi + \rho)$, we may, on writing η for $\pi + \rho$, expand in ascending powers of η, and then substitute $\pi + \rho$ for η in the result; thus

$$\begin{aligned} \sin(\pi + \rho)\, u &= \left(\eta - \frac{\eta^3}{1.2.3} + \&c.\right) u \\ &= \left(\pi + \rho - \frac{(\pi + \rho)^3}{1.2.3} + \&c.\right) u, \end{aligned}$$

but neither by Maclaurin's nor by Taylor's theorem can we obtain an expansion in ascending powers of ρ, unless π and ρ are commutative. The case in which π and ρ combine, according to the law (1), has been already referred to; but to render this analysis more complete, it may be proper to quote the investigation here.

PROP. Let π and ρ be distributive symbols which combine in subjection to the law

$$\rho\, f(\pi)\, u = \lambda f(\pi)\, \rho u \; \ldots\ldots\ldots (2),$$

λ being a functional symbol operating on π in such manner that $\lambda f(\pi) = f\{\phi(\pi)\}$, it is required to expand $f(\pi + \rho)$ in ascending powers of ρ.

We have

$$\left.\begin{aligned} \rho f(\pi)\, u &= \lambda f(\pi)\, \rho u \\ \rho^2 f(\pi)\, u &= \lambda^2 f(\pi)\, \rho^2 u \\ &\vdots \\ \rho^m f(\pi) u &= \lambda^m f(\pi)\, \rho^m u \end{aligned}\right\} \ldots\ldots (3).$$

Let $\pi+\rho=\eta$; then $f(\pi+\rho)u=f(\eta)u$. Now, as η operates solely on u, it is commutative with respect to the constants in $f(\eta)$; wherefore

$$\eta f(\eta)\,u=f(\eta)\,\eta u.$$

Or, dropping the subject u, and writing $\pi+\rho$ for η,

$$(\pi+\rho)f(\pi+\rho)=f(\pi+\rho)(\pi+\rho)\,\ldots\,.(4).$$

Let $f(\pi+\rho)\,u=\Sigma f_m(\pi)\,\rho^m u$; then, still supposing u understood,

$$\begin{aligned}(\pi+\rho)f(\pi+\rho)&=\pi\Sigma f_m(\pi)\,\rho^m+\rho\Sigma f_m(\pi)\,\rho^m\\ &=\Sigma\pi f_m(\pi)\,\rho^m+\Sigma\lambda f_m(\pi)\,\rho^{m+1}\text{ by (3).}\end{aligned}$$

Under the first Σ in the second member the coefficient of ρ^m is $\pi f_m(\pi)$, and under the second Σ the coefficient of ρ^m is $\lambda f_{m-1}(\pi)$; wherefore the aggregate coefficient of ρ^m is

$$\pi f_m(\pi)+\lambda f_{m-1}(\pi)\,\ldots\ldots\ldots\,.(5).$$

Again, we have

$$\begin{aligned}f(\pi+\rho)(\pi+\rho)&=\Sigma f_m(\pi)\,\rho^m\pi+\Sigma f_m(\pi)\,\rho^{m+1}\\ &=\Sigma f_m(\pi)\,\lambda^m\pi\rho^m+\Sigma f_m(\pi)\,\rho^{m+1},\end{aligned}$$

in which the aggregate coefficient of ρ^m is

$$f_m(\pi)\,\lambda^m\pi+f_{m-1}(\pi)\,\ldots\ldots\ldots\,.(6).$$

Equating this with (5), we have

$$f_m(\pi)\,\lambda^m\pi+f_{m-1}(\pi)=\pi f_m(\pi)+\lambda f_{m-1}(\pi),$$

hence
$$f_m(\pi)=\frac{\lambda f_{m-1}(\pi)-f_{m-1}(\pi)}{\lambda^m\pi-\pi};$$

or separating the symbols of operation,

$$f_m(\pi)=\frac{(\lambda-1)f_{m-1}(\pi)}{(\lambda^m-1)\pi}\;\ldots\ldots\,.(7),$$

which expresses the law of formation of the coefficients.

The first term, $f_0(\pi)$, is equal to $f(\pi)$. For let k be a symbol operating on π, in such manner that $kf(\pi)=f_0(\pi)$; then the first term of the expansion of $(\pi+\rho)f(\pi+\rho)$ is $k\pi f(\pi)$: but by (5) this term is $\pi f_0(\pi)=\pi kf(\pi)$; wherefore

$$k\pi f(\pi)=\pi kf(\pi),$$

wherefore π and k are *commutative*. It is hence evident that k can only operate as a constant multiplier, the value of which is independent of the form of $f(\pi)$. Let $f(\pi)=\pi$; then, since $f(\pi+\rho)=\pi+\rho$, it is evident that $k=1$; wherefore

$$f_0(\pi)=f(\pi)$$

in all cases, and the expansion is completely determined.

COR. If the symbols π and ρ combine, according to the law,

$$\rho f(\pi)\, u = f(\pi + \Delta\pi)\, \rho u,$$

$\Delta\pi$ being any constant increment; then

$$f(\pi + \rho) = f(\pi) + \frac{\Delta}{\Delta\pi} f(\pi)\, \rho + \frac{\Delta^2}{\Delta\pi^2} f(\pi)\, \frac{\rho^2}{1.2} + \&c.\,.\,(8),$$

the interpretation of $\dfrac{\Delta}{\Delta\pi}$ being

$$\frac{\Delta}{\Delta\pi} f(\pi) = \frac{f(\pi + \Delta\pi) - f(\pi)}{\Delta\pi} \;\ldots\ldots\; (9),$$

for $\lambda f(\pi) = f(\pi + \Delta\pi)$. Hence $\lambda^m \pi = \pi + m\Delta\pi$, and (7) gives

$$\begin{aligned} f_m(\pi) &= \frac{f_{m-1}(\pi + \Delta\pi) - f_{m-1}(\pi)}{m\Delta\pi} \\ &= \frac{1}{m}\, \frac{\Delta}{\Delta\pi} f_{m-1}(\pi); \end{aligned}$$

whence the theorem is manifest.

If $\Delta\pi$ vanishes, the symbols π and ρ are commutative, $\dfrac{\Delta}{\Delta\pi}$ becomes $\dfrac{d}{d\pi}$, and (8) is reduced to Taylor's theorem.

I proceed now to notice two remarkable deductions from the above theorem, each including several particular results of great interest.

Let us consider the expression $f\left\{x + \phi'\left(\dfrac{d}{dx}\right)\right\} u$, in which $\phi'\left(\dfrac{d}{dx}\right)$ denotes a function of the symbol $\dfrac{d}{dx}$, derived from a certain other arbitrary function $\phi\left(\dfrac{d}{dx}\right)$, in like manner as $\phi'(t) = \dfrac{d}{dt}\phi(t)$.

Now $\phi'\left(\dfrac{d}{dx}\right)$ is the limit to which $\phi'\left(\theta + \dfrac{d}{dx}\right)$ approaches as θ approximates to 0. Hence $f\left\{x + \phi'\left(\dfrac{d}{dx}\right)\right\} u$ is the limit of $f\left\{x + \phi'\left(\theta + \dfrac{d}{dx}\right)\right\} u$. But

$$\begin{aligned} f\left\{x + \phi'\left(\theta + \frac{d}{dx}\right)\right\} u &= f\{x + \varepsilon^{\frac{d}{dx}\frac{d}{d\theta}} \phi'(\theta)\}\, u \\ &= f(x + \rho)\, u, \end{aligned}$$

if $\varepsilon^{\frac{d}{dx}\frac{d}{d\theta}} \phi'(\theta) = \rho$.

Now, since

$$\varepsilon^{\frac{d}{dx}\frac{d}{d\theta}}\phi'(\theta)f(x)\,u = \varepsilon^{\frac{d}{d\theta}\frac{d}{dx}}f(x)\,\phi'(\theta)\,u$$
$$= f\left(x+\frac{d}{d\theta}\right)\varepsilon^{\frac{d}{d\theta}\frac{d}{dx}}\phi'(\theta)\,u,$$

we have, on substitution of ρ,

$$\rho f(x)\,u = f\left(x+\frac{d}{d\theta}\right)\rho u.$$

Hence, if in (8) we write x in the place of π, and suppose $\Delta x = \frac{d}{d\theta}$, we have

$$f(x+\rho)\,u = \left\{f(x) + \frac{\Delta}{\Delta x}f(x)\,\rho + \frac{1}{1.2}\,\frac{\Delta^2}{\Delta x^2}f(x)\,\rho^2 + \&c.\right\}u$$
$$= \varepsilon^{\frac{\Delta'}{\Delta x}\rho''}f(x)\,u \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

in which the symbol $\frac{\Delta'}{\Delta x}$ is accented to indicate that it refers to $f(x)$ only, and ρ'' is doubly accented to shew that it refers to u only.

Now, since $\Delta x = \frac{d}{d\theta}$, we have by (9),

$$\frac{\Delta'}{\Delta x} = \frac{\varepsilon^{\frac{d}{d\theta}\frac{d'}{dx}} - 1}{\frac{d}{d\theta}} = \left(\varepsilon^{\frac{d}{d\theta}\frac{d'}{dx}} - 1\right)\left(\frac{d}{d\theta}\right)^{-1};$$

moreover $\qquad \rho'' = \varepsilon^{\frac{d''}{dx}\frac{d}{d\theta}}\phi'(\theta).$

Substituting these values in (10), we find

$$f(x+\rho)\,u = \varepsilon^{\left(\varepsilon^{\frac{d}{d\theta}\frac{d'}{dx}}-1\right)\left(\frac{d}{d\theta}\right)^{-1}\varepsilon^{\frac{d''}{dx}\frac{d}{d\theta}}\phi'(\theta)}f(x)\,u$$
$$= \varepsilon^{\left\{\varepsilon^{\left(\frac{d'}{dx}+\frac{d''}{dx}\right)\frac{d}{d\theta}} - \varepsilon^{\frac{d''}{dx}\frac{d}{d\theta}}\right\}\left(\frac{d}{d\theta}\right)^{-1}\phi'(\theta)}f(x)\,u$$
$$= \varepsilon^{\left(\varepsilon^{\frac{d}{dx}\frac{d}{d\theta}} - \varepsilon^{\frac{d''}{dx}\frac{d}{d\theta}}\right)\phi(\theta)}f(x)\,u \ldots\ldots (11).$$

Since $\frac{d'}{dx} + \frac{d''}{dx} = \frac{d}{dx}$, to be taken with reference to both $f(x)$ and u; and since $\frac{d^{-1}}{d\theta}\phi'(\theta) = \phi(\theta)$. But

$$\left(\varepsilon^{\frac{d}{dx}\frac{d}{d\theta}} - \varepsilon^{\frac{d''}{dx}\frac{d}{d\theta}}\right)\phi(\theta) = \phi\left(\theta+\frac{d}{dx}\right) - \phi\left(\theta+\frac{d''}{dx}\right);$$

whence $\quad f(x+\rho)\,u = \varepsilon^{\phi\left(\theta+\frac{d}{dx}\right)-\phi\left(\theta+\frac{d''}{dx}\right)} f(x)\,u.$

Let θ vanish, and putting for ρ its limiting value $\phi'\left(\frac{d}{dx}\right)$, we have

$$f\left\{x+\phi'\left(\frac{d}{dx}\right)\right\}u = \varepsilon^{\phi\left(\frac{d}{dx}\right)-\phi\left(\frac{d''}{dx}\right)} f(x)\,u.$$

As, however, $\varepsilon^{-\phi\left(\frac{d''}{dx}\right)}$ does not affect $f(x)$, we may transpose it, and remove the double accent; whence

$$f\left\{x+\phi'\left(\frac{d}{dx}\right)\right\}u = \varepsilon^{\phi\left(\frac{d}{dx}\right)} f(x)\,\varepsilon^{-\phi\left(\frac{d}{dx}\right)} u \ldots . (12),$$

which is the first of the results in question.

Considering, secondly, the expression $f\left\{\frac{d}{dx}+\phi'(x)\right\}u$, if we write this in the form

$$f\left\{\frac{d}{dx}+\phi'(\theta+x)\right\}u = f\left\{\frac{d}{dx}+\varepsilon^{x\frac{d}{d\theta}}\phi'(\theta)\right\}u,$$

and proceed as above, we finally get

$$f\left\{\frac{d}{dx}+\phi'(x)\right\}u = \varepsilon^{-\phi(x)} f\left(\frac{d}{dx}\right)\varepsilon^{\phi(x)}\,u \ldots . (13),$$

which is the second of the theorems in question. Perhaps this result might be obtained more simply by induction. We shall now notice a few applications.

In the particular case in which $f\left\{\frac{d}{dx}+\phi'(x)\right\}$ is of the form $\left\{\frac{d}{dx}+\phi'(x)\right\}^{-1}$, we have, by (13),

$$\left\{\frac{d}{dx}+\phi'(x)\right\}^{-1}u = \varepsilon^{-\phi(x)}\left(\frac{d}{dx}\right)^{-1}\varepsilon^{\phi(x)}\,u \ldots . (14),$$

which is the known solution of the linear differential equation of the first order.

If, in (12), $\phi'\left(\frac{d}{dx}\right)$ be of the form $\frac{d^{-1}}{dx}$, the expansion will stop at the second term, whatever may be the form of the function denoted by f.

For $\quad f\left\{x+\left(\frac{d}{dx}\right)^{-1}\right\}u = \varepsilon^{\log\frac{d}{dx}} f(x)\,\varepsilon^{-\log\frac{d}{dx}}u$

$$= \frac{d}{dx}f(x)\left(\frac{d}{dx}\right)^{-1}u$$

$$= f'(x)\,u + f(x)\int u\,dx \;. \;\; . (15).$$

In like manner the expansion of $f\left(\frac{d}{dx}+\frac{1}{x}\right)u$ stops at the second term; thus

$$f\left(\frac{d}{dx}+\frac{1}{x}\right)u=\varepsilon^{-\log x}f\left(\frac{d}{dx}\right)\varepsilon^{\log x}u$$

$$=\frac{1}{x}f\left(\frac{d}{dx}\right)xu$$

$$=f\left(\frac{d}{dx}\right)u+\frac{1}{x}f'\left(\frac{d}{dx}\right)u.\ldots(16).$$

In (12) let $\phi'\left(\frac{d}{dx}\right)=\frac{d}{dx}$, we have

$$f\left(x+\frac{d}{dx}\right)u=\varepsilon^{\frac{1}{2}\frac{d^2}{dx^2}}f(x)\,\varepsilon^{-\frac{1}{2}\frac{d^2}{dx^2}}u$$

$$=\varepsilon^{\frac{1}{2}\left(\frac{d'}{dx}+\frac{d''}{dx}\right)^2-\frac{1}{2}\frac{d''^2}{dx^2}}f(x)\,u,$$

in which $\frac{d'}{dx}$ refers to $f(x)$, and $\frac{d''}{dx}$ to u. Hence

$$f\left(x+\frac{d}{dx}\right)u=\varepsilon^{\frac{1}{2}\frac{d'^2}{dx^2}+\frac{d'}{dx}\frac{d''}{dx}}f(x)\,u$$

$$=\varepsilon^{\frac{1}{2}\frac{d'^2}{dx^2}}\left\{f(x)+f'(x)\frac{d}{dx}+\frac{1}{1.2}f''(x)\frac{d^2}{dx^2}+\&c.\right\}u.$$

Let $\varepsilon^{\frac{1}{2}\frac{d^2}{dx^2}}f(x)=f_0(x)$; then

$$f\left(x+\frac{d}{dx}\right)u=\{f_0(x)+f_0'(x)\frac{d}{dx}+\frac{1}{1.2}f_0''(x)\frac{d^2}{dx^2}+\&c.\}\,u\ldots(17).$$

The coefficients of the expansion, after the first term, follow the law of Taylor's theorem, which is a remarkable circumstance, seeing that the symbols x and $\frac{d}{dx}$ are not commutative

In (13) let $\phi'(x)=x$, we have

$$f\left(\frac{d}{dx}+x\right)u=\varepsilon^{-\frac{1}{2}x^2}f\left(\frac{d}{dx}\right)\varepsilon^{\frac{1}{2}x^2}u.$$

From this equation, after a troublesome reduction, I find

$$f\left(\frac{d}{dx}+x\right)u=\left\{f_0\left(\frac{d}{dx}\right)+f_0'\left(\frac{d}{dx}\right)x+\frac{1}{1.2}f_0''\left(\frac{d}{dx}\right)x^2+\&c.\right\}u \quad\ldots\ldots(18),$$

in which $f_0\left(\frac{d}{dx}\right)$ is formed from $f\left(\frac{d}{dx}\right)$ in the same way

as $f_0(x)$ is formed from $f(x)$ in the preceding expansion. The developments are thus seen to be of precisely the same form, which again is a remarkable circumstance.

Some of the above deductions may be applied to the solution of differential equations. Thus, if we have an equation of the form

$$f_0(x)\,u + f_0'(x)\frac{du}{dx} + \frac{1}{1.2}f_0''(x)\frac{du^2}{dx^2} + \&c. = 0\ldots(19),$$

in which $f_0(x)$ is a rational and integral function of x, we may at once place it under the form

$$f\left(x + \frac{d}{dx}\right)u = 0,$$

the form of the function denoted by f being determined by the relation

$$f(x) = \varepsilon^{-\frac{1}{2}\frac{d^2}{dx^2}}f_0(x)\ldots\ldots\ldots(20).$$

Let $x + \dfrac{d}{dx} = \lambda$; then, supposing a_1, a_2.... to be the roots of the equation $f(x) = 0$, we have a system of equations of the form

$$(\lambda - a_1)\,u = 0 \quad \text{or} \quad (x - a_1)\,u + \frac{du}{dx} = 0,$$

$$(\lambda - a_2)\,u = 0 \quad \text{or} \quad (x - a_2)\,u + \frac{du}{dx} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

If $u_1 = 0$, $u_2 = 0$, ... are the particular integrals thus obtained, then

$$u = c_1u_1 + c_2u_2 + \&c.$$

will be the complete integral.

Ex. Given $(x^2 - 5x + 7)\,u + (2x - 5)\dfrac{du}{dx} + \dfrac{d^2u}{dx^2} = 0.$

Here $f_0(x) = x^2 - 5x + 7$; whence

$$f(x) = \varepsilon^{-\frac{1}{2}\frac{d^2}{dx^2}}f_0(x) = \left(1 - \frac{1}{2}\frac{d^2}{dx^2} + \&c.\right)f_0(x)$$

$$= x^2 - 5x + 6$$

$$= (x - 2)(x - 3);$$

we have therefore the system

$$(\lambda - 2)\,u = 0, \quad \text{or} \quad (x - 2)\,u + \frac{du}{dx} = 0.$$

$$(\lambda - 3)\,u = 0, \quad \text{or} \quad (x - 3)\,u + \frac{du}{dx} = 0,$$

therefore $$u = c_1\varepsilon^{2x - \frac{x^2}{2}} + c_2\varepsilon^{3x - \frac{x^2}{2}}.$$

By (16) we may in like manner integrate any equation of the form

$$xf\left(\frac{d}{dx}\right)u+f'\left(\frac{d}{dx}\right)u=0\ldots\ldots\ldots(21).$$

There are some other equations, particularly in finite differences, which the above theorems enable us to solve. It is only however in connexion with the *symbolical* form of the linear equation, as discussed in the paper above referred to, that such applications can be reduced to a uniform and general system.

It might have been thought simpler, in the preceding investigations, to apply directly the principle by which the expansion of $f(\pi+\rho)$ was obtained, and thus to deduce the expansions of $f\left(x+\frac{d}{dx}\right)$ from the fundamental equation

$$\left(x+\frac{d}{dx}\right)f\left(x+\frac{d}{dx}\right)u+f\left(x+\frac{d}{dx}\right)\left(x+\frac{d}{dx}\right)u.$$

This method would indeed at once have given the law of derivation of the coefficients *after the first*, but it would still have been necessary, in order to obtain the first term, to adopt a process of reasoning similar to that which we have in reality employed. In illustration of this remark, let us take the expression $f\left(u+v\frac{d}{dx}\right)U$, u and v being functions of x, and let us seek to expand this in the form $\Sigma f_m(x)\frac{d^m}{dx}U$. We have

$$\left(u+v\frac{d}{dx}\right)f\left(u+v\frac{d}{dx}\right)U=f\left(u+v\frac{d}{dx}\right)\left(u+v\frac{d}{dx}\right)U,$$

or $\left(u+v\frac{d}{dx}\right)\Sigma f_m(x)\left(\frac{d}{dx}\right)^m U=\Sigma f_m(x)\left(\frac{d}{dx}\right)^m\left(u+v\frac{d}{dx}\right)U\ldots(22).$

Now the first member becomes

$$\Sigma\left\{uf_m(x)\left(\frac{d}{dx}\right)^m+vf_m'(x)\left(\frac{d}{dx}\right)^m+vf_m(x)\left(\frac{d}{dx}\right)^{m+1}\right\}U,$$

in which the aggregate coefficient of $\left(\frac{d}{dx}\right)^m$ is

$$uf_m(x)+vf_m'(x)+vf_{m-1}(x)\ldots\ldots\ldots\ldots(23).$$

Similarly we find, as the coefficient of $\left(\frac{d}{dx}\right)^m$ in the second member, the series

$$f_{m-1}(x) + f_m(x)(u + mv') + (m+1) f_{m+1}(x)\left(u' + \frac{m}{1.2} v''\right)$$
$$+ (m+2) f_{m+2}(x)\left\{u'' + \frac{m(m+1)}{1.2.3} v'''\right\} + \&c.$$

And, equating these results,

$$vf'_m(x) - mv'f_m(x) = (m+1) f_{m+1}(x)\left(u' + \frac{m}{1.2} v''\right)$$
$$+ (m+2) f_{m+2}(x)\left(u'' + \frac{m(m+1)}{1.2.3} v'''\right) + \&c. \dots (24),$$

which is the law connecting the coefficients of the expansion.

In the case of $f\left(x + \frac{d}{dx}\right)$, we have $u = x$, $v = 1$, $u' = 1$, $v' = 0$, $u'' = 0$, &c.; whence, substituting in (24), we have

$$f'_m(x) = (m+1) f_{m+1}(x),$$
$$\text{or } f_{m+1}(x) = \frac{1}{m+1} f_m'(x).$$

This agrees with our previous result, but it leaves $f_0(x)$ undetermined.

In the case of $f\left(x\frac{d}{dx}\right) U$, we have $u = 0$, $v = x$, $v' = 1$, whence

$$xf_m'(x) - mf_m(x) = 0,$$

and, solving the equation,

$$f_m(x) = cx^m.$$

This shews that the expansion of $f\left(x\frac{d}{dx}\right) U$ is of the form $\Sigma a_m x^m \left(\frac{d}{dx}\right)^m U$, as is perhaps known.

In a subsequent part, it is intended to apply the method of this paper to the development of polynomials.

Lincoln, Jan. 3, 1845.

IV.—DEMONSTRATION OF A FUNDAMENTAL PROPOSITION IN THE MECHANICAL THEORY OF ELECTRICITY.

By William Thomson, B.A. St. Peter's College.

If a material point be in a position of equilibrium when under the influence of any number of masses attracting it or repelling it with forces which are inversely proportional to the square of the distance, the equilibrium will be unstable.*

* This theorem was first given by Mr. Earnshaw, in his Memoir on Molecular Forces, read at the Cambridge Philosophical Society, March 18, 1839. See vol. VII. of the *Transactions.*

The first thing to be proved is, that if the material point receive a slight displacement, there will in general be a moving force called into action.

Let O be the position of equilibrium; P any adjacent point; V the potential of the influencing masses, μ, at P, which point we suppose not to be contained within any portion of μ; U the value of V at O. Now it is shown by Gauss, in his Mémoire on General Theorems in Attraction, that V cannot have the constant value U through any finite volume, however small, adjacent to O, without having it for every point external to μ. But this is impossible, as may be shown in the following manner.

Let σ be a closed surface containing within it a quantity of matter, μ_1, consisting of any number of detached portions of μ, or of the whole of μ, if μ be a continuous mass. Let $d\sigma$ be an element of σ, and P the force due to the total action of μ, resolved in a direction perpendicular to $d\sigma$, which may be considered positive when directed towards the space within σ. Then, by a theorem demonstrated in this *Journal* (vol. III. p. 203), we have

$$\iint P d\sigma = 4\pi\mu,$$

the integrations being extended over the whole of σ. Hence P cannot be $=0$ for every point of the surface σ, and therefore V cannot be constant for all the space exterior to μ.

Hence V cannot have the constant value U for every point of any finite volume, however small, adjacent to O.

Now let a sphere S be described round O as centre, with any radius a, sufficiently small that no portion of μ shall be included, and let P be any point of the surface S, and ds an element of the surface at P.

In the equations (3) and (4) of the article already referred to (vol. III. p. 202), let the sphere S be the surface there considered; let $v = V$, and $v_1 = \frac{1}{r}$, if $OP = r$.

Hence $P_1 = \frac{1}{a^2}$, $v_1 = \frac{1}{a}$, at every point of S;

$$m = \mu, \quad m_1 = 1, \quad \iiint v dm_1 = U.$$

Also $$\iint v_1 P ds = \frac{1}{a} \iint P ds = 0,$$

and $\iiint v_1 dm = 0$, since S does not contain any of the matter μ. We have therefore, by comparing (3) and (4),

$$0 = 4\pi U - \frac{1}{a^2} \iint V ds.$$

Therefore $$\iint V ds = 4\pi a^2 U,$$
which shows that the mean value for the surface of a sphere, of the potential of any external masses, is equal to the value at the centre. Let $V = U + u$.

Therefore $$\iint u ds = 0.$$

Now, as has already been shown, u cannot be $= 0$ for every point P adjacent to O, and therefore if the sphere pass through a point P' where u is negative, there must also be a point P'' in the surface, for which u is positive. But if we assume the potential of an attracting particle to be positive, the direction of the resultant force, resolved along any straight line, will be that in which V increases. Hence there will be a force towards O, for points displaced along OP', and from O, for points displaced along OP''. Hence if M, the material point in equilibrium at O, be displaced along OP'', the moving force generated will tend to remove it further from O, which is therefore an unstable position.

As an application of this theorem, let us consider the case of any number of material points repelling one another according to the inverse square of the distance, and contained in the interior of a rigid closed envelope. Let the system be in equilibrium when acted upon by attracting or repelling masses distributed in any manner without the envelope.

It will generally be possible that there may be a position or positions of equilibrium, in which at least some of the particles are not in contact with the surface. If now we suppose all the particles fixed except one, not in contact with the surface, the equilibrium of this particle is, as has been shown, unstable. Hence, generally, the equilibrium of the system is unstable if any of the particles be not in contact with the surface, and therefore in nature the particles cannot remain in such a position. There must, however, be some stable position or positions in which the particles can rest, but in such, all the particles must be in contact with the surface of the envelope. The sole condition of equilibrium in this case will be that the resultant force on each particle shall be in the direction of a normal to the surface, and directed towards the exterior space. If the number of particles be infinite, and there be one position in which the whole surface is covered, there can be no other in which this is the case, as is shown in the paper in this *Journal* already quoted (vol. III. p. 205); and it is also readily seen that this position will be stable, and that no other in which the surface is not entirely covered can be stable. In this case the particles

will be distributed according to the law of the intensity of electricity on the surface, the space within being conducting matter, and the masses without being any electrified bodies. If a mechanical theory be adopted, *electricity* will actually be a number of material points without weight, which repel one another according to the inverse square of the distance. Thus the result we have arrived at is, that there can be permanently no free electricity in the interior of a conducting body under any circumstances whatever.

If, as may happen through the influence of the exterior masses, there cannot be a position of equilibrium of the particles covering the whole surface, there will be a permanent distribution, in which part of the surface is uncovered. This however is never the case with electricity, as a certain quantity of latent electricity is then decomposed, so that the whole surface is covered with electricity, either positive or negative. All the above reasoning would still apply, if we considered the masses of some points to be negative, and of some positive, and the force between any two be a repulsion equal to the product of their masses divided by the square of their distance.

Since every particle is on the surface, the whole *medium* (if it can be properly so called), will be an indefinitely thin stratum, the thickness being in fact the ultimate breadth of an atom or material point. If we suppose these atoms to be merely centres of force, the thickness will therefore be absolutely nothing, and thus the *fluid* will be absolutely compressible and inelastic. Any thickness which the stratum can have must depend on a force of elasticity, or on a force generated by the contact of material points, and in either case will therefore require an ultimate law of repulsion more intense than that of the inverse square,* when the distance is very small, and we therefore conclude that this cannot be the ultimate law of repulsion in any elastic fluid. As, however, all experiments yet made serve to confirm the fact that there is no electricity in the interior of conducting bodies, or that the stratum has absolutely no thickness, we conclude that there is no elasticity in the assumed electric fluid, and thus the law of force, deduced independently by direct experiments, is confirmed.

* This agrees with a result of Mr. Earnshaw.

St. Peter's College, Jan. 16, 1845.

VI.—ON THE REDUCTION OF THE GENERAL EQUATION OF SURFACES OF THE SECOND ORDER.

By William Thomson, B.A., St. Peter's College.

In the following paper, by a simple assumption with reference to the coefficients in the general equation of a surface of the second order, the cubic, by means of which the three principal axes are determined is made to assume a very simple form, which enables us to prove the reality of the roots, and to find the limits between which they lie, with great ease. It also leads to a very simple analytical proof, that the three principal axes are at right angles to one another.

Let
$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy + 2A''x + 2B''y + 2C''z = H,$$

or, for brevity $H_2(x, y, z) + H_1(x, y, z) = H$,

be the equation to a surface of the second order.

Let
$$\left.\begin{array}{lll} A' = (gh)^{\frac{1}{2}}, & B' = (hf)^{\frac{1}{2}}, & C' = (fg)^{\frac{1}{2}}, \\ A = f + \alpha, & B = g + \beta, & C = h + \gamma, \end{array}\right\} \dots . (1),$$

from which we deduce

$$\left.\begin{array}{lll} f = \dfrac{B'C'}{A'}, & g = \dfrac{C'A'}{B'}, & h = \dfrac{A'B'}{C'}, \\ \alpha = A - \dfrac{B'C'}{A'}, & \beta = B - \dfrac{C'A'}{B'}, & \gamma = C - \dfrac{A'B'}{C'} \end{array}\right\} \dots (2);$$

which express real determinate values for f, g, &c. in terms of the given coefficients. Making the substitutions (1), we find
$$H_2 = \alpha x^2 + \beta y^2 + \gamma z^2 + (f^{\frac{1}{2}}x + g^{\frac{1}{2}}y + h^{\frac{1}{2}}z)^2 \dots . (a).$$

Now we may define a principal axis to be a line such that, if it be taken for the axis of x', and the axes of y' and z' be in the plane perpendicular to it, the products $x'y'$ and $x'z'$ shall disappear in the transformed expression H_2. This will be ensured if the product $x'y'$ vanishes for every point in the plane $x'y'$, and for every position of this plane passing through the principal axis OX'; a definition equivalent to the one in which a principal axis is defined as an axis which is perpendicular to its diametral plane.

Let l, m, n be the direction-cosines of a principal axis OX'; l', m', n' those of any line OY' perpendicular to it; x', y' the co-ordinates of any point in the plane $X'OY'$;

x, y, z the co-ordinates of the same point referred to the original axes: we have

$$x = lx' + l'y',$$
$$y = mx' + m'y',$$
$$z = nx' + n'y'.$$

Therefore $\alpha x^2 + \beta y^2 + \gamma z^2 + (f^{\frac{1}{2}}x + g^{\frac{1}{2}}y + h^{\frac{1}{2}}z)^2$

$$= (\alpha l^2 + \beta m^2 + \gamma n^2)\, x'^2 + (\alpha l'^2 + \beta m'^2 \gamma n'^2)\, y'^2$$
$$+ 2\,(\alpha ll' + \beta mm' + \gamma nn')\, x'y'$$
$$+ \{(f^{\frac{1}{2}}l + g^{\frac{1}{2}}m + hn)\, x' + (f^{\frac{1}{2}}l' + g^{\frac{1}{2}}m' + h^{\frac{1}{2}}n')\, y'\}^2;$$

which becomes $\qquad Px'^2 + P'y'^2,$

if we put for brevity

$$S = f^{\frac{1}{2}}l + g^{\frac{1}{2}}m + h^{\frac{1}{2}}n \;\ldots\ldots\ldots\; (3),$$
$$P = S^2 + \alpha l^2 + \beta m^2 + \gamma n^2 \;\ldots\ldots\; (4),$$

and similarly for $l'm'n'$, and if we assume the coefficient of $x'y' = 0$, which gives

$$(Sf^{\frac{1}{2}} + \alpha l)\, l' + (Sg^{\frac{1}{2}} + \beta m)\, m' + (Sh^{\frac{1}{2}} + \gamma n)\, n' = 0.$$

If OX' be a principal axis, this must hold for all values of l', m', n' consistent with

$$ll' + mm' + nn' = 0;$$

we must therefore have

$$\frac{Sf^{\frac{1}{2}} + \alpha l}{l} = \frac{Sg^{\frac{1}{2}} + \beta m}{m} = \frac{Sh^{\frac{1}{2}} + \gamma n}{n}:$$

therefore each member is

$$= l\,(Sf^{\frac{1}{2}} + \alpha l) + m\,(Sf^{\frac{1}{2}} + \beta m) + n\,(Sf^{\frac{1}{2}} + \gamma n),$$
$$= S^2 + \alpha l^2 + \beta m^2 + \gamma n^2,$$
$$= P.$$

Therefore
$$\left.\begin{aligned} l &= \frac{Sf^{\frac{1}{2}}}{P - \alpha}, \\ m &= \frac{Sg^{\frac{1}{2}}}{P - \beta}, \\ n &= \frac{Sh^{\frac{1}{2}}}{P - \gamma}, \end{aligned}\right\} \ldots\ldots\ldots\ldots (5).$$

Hence, by (3),

$$S\left(\frac{f}{P - \alpha} + \frac{g}{P - \beta} + \frac{h}{P - \gamma} - 1\right) = 0;$$

and therefore, unless $S = 0$, which, on account of (5), would

require $P = \alpha = \beta = \gamma$, a case that will be considered below, we must have

$$\frac{f}{P-\alpha} + \frac{g}{P-\beta} + \frac{h}{P-\gamma} - 1 = 0 \ldots\ldots (6),$$

which determines P. This equation, being a cubic, gives three values for P. Now, from equations (2) it follows that f, g, h must either be all positive or all negative; the former being the case when two or more of A', B', C', and the latter when one or three are negative.* Hence, if α, β, γ be in descending order of magnitude, and e be an indefinitely small quantity, and if we substitute

$$\alpha - e, \quad \beta + e, \quad \text{and} \quad \beta - e, \quad \gamma + e,$$

for P in the first member of (6), the first and second values will have contrary signs, and so will the third and fourth. Hence the cubic has a real root between α, β, and another between β, γ, and its remaining root must therefore also be real, and between ∞ and α, or between γ and $-\infty$. The first is obviously the case when f, g, and h are positive, and the second when they are negative.

Let P_1, P_2 be any two roots of (6), and let l_1, m_1, n_1, l_2, m_2, n_2 be the corresponding values of l, m, n deduced from equations (5). Writing down (6) for each value, and subtracting, we have

$$(P_1 - P_2)\left\{\frac{f}{(P_1-\alpha)(P-\alpha)} + \frac{g}{(P_2-\beta)(P_2-\beta)} + \frac{h}{(P_1-\gamma)(P_2-\gamma)}\right\} = 0.$$

If P_1 be different from P_2, the second factor must vanish, or, by (5), $$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \ldots\ldots\ldots (b).$$

Hence any two of the axes determined by equations (6) and (5) are at right angles, and therefore the three must form a rectangular system. If we take it for axes of co-ordinates, and if P, Q, R be the three roots of (6), the equation to the surface becomes $Px^2 + Qy^2 + Rz^2 + H_1 = H$,

accents being omitted. If none of the quantities P, Q, R vanishes, we may obviously, by changing the origin, make H_1 disappear, and the equation will be reduced to the form

$$Px^2 + Qy^2 + Rz^2 = H',$$

which is the equation of surfaces of the second order referred to principal axes through the centre.

* Hence (6) is a particular case of a certain equation of any order, which has been shown by M. M. Plana and Liouville to have all its roots real. See Moigno, *Calc. Int.* p. 296.

If in this equation H' be $=0$, the surface represented will be either a point or a cone, according as P, Q, and R have the same or different signs. Excluding this case, we may, without losing generality, consider H' as positive.

The equation will then represent an ellipsoid if P, Q, R be all positive; a hyperboloid of one sheet if one of them only be negative, and a hyperboloid of two sheets if two of them be negative. If all three be negative, the surface will be imaginary.

Hence, from equation (6) we infer that if f, g, h, α, β, γ, be all positive the surface will be an ellipsoid, and if they be all negative it will be imaginary.

In addition to these we have the following cases.

I. $\dfrac{f}{-\alpha} + \dfrac{g}{-\beta} + \dfrac{h}{-\gamma} - 1 > 0.$

(1) $fgh > 0.$

$\alpha > 0$, $\beta > 0$, $\gamma < 0$, Ellipsoid,
$\alpha > 0$, $\beta < 0$, $\gamma < 0$, Hyperboloid of one sheet,
$\alpha < 0$, $\beta < 0$, $\gamma < 0$, Hyperboloid of two sheets,

(2) $fgh < 0.$

$\alpha > 0$, $\beta > 0$, $\gamma > 0$, Hyperboloid of one sheet,
$\alpha > 0$, $\beta > 0$, $\gamma < 0$, Hyperboloid of two sheets,
$\alpha > 0$, $\beta < 0$, $\gamma < 0$, Imaginary.

II. $\dfrac{f}{-\alpha} + \dfrac{g}{-\beta} + \dfrac{h}{-\gamma} - 1 < 0.$

(1) $fgh > 0.$

$\alpha > 0$, $\beta > 0$, $\gamma < 0$, Hyperboloid of one sheet,
$\alpha > 0$, $\beta < 0$, $\gamma < 0$, Hyperboloid of two sheets,
$\alpha < 0$, $\beta < 0$, $\gamma < 0$, Imaginary.

(2) $fgh < 0.$

$\alpha > 0$, $\beta > 0$, $\gamma > 0$, Ellipsoid,
$\alpha > 0$, $\beta > 0$, $\gamma < 0$, Hyperboloid of one sheet,
$\alpha > 0$, $\beta < 0$, $\gamma < 0$, Hyperboloid of two sheets.

These tests enable us, when A, B, C, A', B, C' are given numerically, to find the nature of the surface represented, provided it has a centre, by calculating α, β, γ, f, g, h from equations (2).

If the surface have not a centre, it can belong to neither of the four cases considered above, and we must therefore have

$$\frac{f}{\alpha} + \frac{g}{\beta} + \frac{h}{\gamma} + 1 = 0,$$

which is the condition that one root of (6) may be $=0$. If we substitute for α, β, γ, their values, by (1), and clear of fractions, this becomes

$$ABC + 2fgh - Agh - Bhf - Cfg = 0,$$

or $$ABC + 2A'B'C' - AA'^2 - BB'^2 - CC'^2 = 0,$$

which agrees with the condition given in Gregory's *Solid Geometry*, p. 64.

This is the condition that must be satisfied in the case in which it is impossible to make H_1 vanish, from the general equation, by any finite change in the position of the origin, as may be readily verified.

If the surface be of revolution, two of the roots of (6) must be equal. Hence each of the two must be equal to one of the quantities α, β, γ, on account of the limits found above for the roots. Hence, clearing of fractions, we find for the conditions

$$\alpha = \beta = \gamma,$$

and the remaining root will be given by

$$f + g + h - (P - \alpha) = 0.$$

Hence, restoring the original constants, we have, in the case of surfaces of revolution,

$$A - \frac{B'C'}{A'} = B - \frac{C'A'}{B'} = C - \frac{A'B'}{C'} = Q = R,$$

which agrees with the condition given by *Gregory*, p. 109,

and $$P = Q + \frac{B'C'}{A'} + \frac{C'A'}{B'} + \frac{A'B'}{C'},$$

or $$= A + B + C - 2Q.$$

The formulæ which have been proved above furnish us with a very simple proof of the following theorem of Chasles.

The principal axes of a cone touching a surface of the second order, are perpendicular to the three confocal surfaces of the second order which intersect in the vertex.

Let $$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

be the equation to the surface. That of the tangent cone through ξ, η, ζ is

$$\left(\frac{\xi^2}{a}+\frac{\eta^2}{b}+\frac{\zeta^2}{c}-1\right)\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1\right)=\left(\frac{\xi x}{a}+\frac{\eta y}{b}+\frac{\zeta z}{c}-1\right)^2.$$

Here $$H_2=\left(\frac{\xi x}{a}+\frac{\eta y}{b}+\frac{\zeta z}{c}\right)^2-K\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right),$$

where $$K=\frac{\xi^2}{a}+\frac{\eta^2}{b}+\frac{\zeta^2}{c}-1.$$

Hence, comparing with (a), we have

$$f=\frac{\xi^2}{a^2},\quad g=\frac{\eta^2}{b^2},\quad h=\frac{\zeta^2}{c^2},$$

$$\alpha=-\frac{K}{a},\quad \beta=-\frac{K}{b},\quad \gamma=-\frac{K}{c}.$$

Therefore (6) becomes

$$\frac{\xi^2}{a(aP+K)}+\frac{\eta^2}{b(bP+K)}+\frac{\zeta^2}{c(cP+K)=1};$$

but $$\frac{\xi^2}{aK}+\frac{\eta^2}{bK}+\frac{\zeta^2}{cK}=1+\frac{1}{K}.$$

Hence, by subtraction,

$$\frac{\xi^2}{a+\frac{K}{P}}+\frac{\eta^2}{b+\frac{K}{P}}+\frac{\zeta^2}{c+\frac{K}{P}}=1 \quad \ldots\ldots\ldots (c).$$

Also, equations (5) give

$$\frac{l\left(a+\frac{K}{P}\right)}{\xi}=\frac{m\left(b+\frac{K}{P}\right)}{\eta}=\frac{n\left(c+\frac{K}{P}\right)}{\zeta}.$$

Hence l, m, n, are the direction-cosines of a normal at ξ, η, ζ, to the surface represented by (c) when either of the three values of $\frac{K}{P}$ is used in that equation. If the value $>a$ be used, the surface represented will be the ellipsoid confocal with the given surface which passes through the vertex of the cone; the value between a and b corresponds to the confocal hyperboloid of one sheet; and the value between b and c to the confocal hyperboloid of two sheets. Thus we infer that these three surfaces intersect at right angles in the

vertex of the cone, and that the three principal axes of the cone touch their lines of intersection.

The same theorem might be proved separately for the different cases of surfaces without centres; but, as these are only extreme cases of central surfaces, we may infer the truth of the theorem for them, as whatever is true in general, is true in limiting cases, provided the result remain definite.

St. Peter's College, Jan. 11, 1845.

V.—ON CERTAIN INTEGRAL TRANSFORMATIONS.

By B. BRONWIN.

THIS paper is a continuation of one bearing the same title printed in the 12th number of this *Journal*, and the references are made to the transformations there given when the number is below (21).

$$\text{Let } y = \frac{(1+b)\,x\sqrt{(1-x^2)}}{\sqrt{(1-c^2x^2)}}, \quad b = \sqrt{(1-c^2)}, \quad k = \frac{1-b}{1+b},$$

$$\text{or } c = \frac{2k^{\frac{1}{2}}}{1+k} \ldots (f).$$

We find $\sqrt{(1-y^2)} = \dfrac{1-(1+b)x^2}{\sqrt{(1-c^2x^2)}}$, $\sqrt{(1-k^2y^2)} = \dfrac{1-(1-b)x^2}{\sqrt{(1-c^2x^2)}}$,

$$\sqrt{(1-y^2)}.\sqrt{(1-k^2y^2)} = \frac{1-2x^2+c^2x^4}{1-c^2x^2}, \quad \frac{\sqrt{(1-y^2)}}{\sqrt{(1-k^2y^2)}} = \frac{1-(1+b)x^2}{1-(1-b)x^2},$$

$$\sqrt{(1-c^2x^2)} = \frac{k\sqrt{(1-y^2)} + \sqrt{(1-k^2y^2)}}{1+k},$$

$$2x^2 = 1 + ky^2 - \sqrt{(1-y^2)}\sqrt{(1-k^2y^2)};$$

$$\frac{dx}{\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}} = \frac{1+k}{2}\,\frac{dy}{\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}}.$$

Multiplying the last by any combination of those which precede it (and the resulting form will shew the combination employed), we easily obtain

$$\frac{x^{\frac{1}{2}}dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{3}{4}}} = \left(\frac{1+k}{2}\right)^{\frac{3}{2}} \frac{y^{\frac{1}{2}}dy}{\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}} \ldots (21).$$

$$\frac{dx}{x^{\frac{1}{2}}(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{4}}} = \left(\frac{1+k}{2}\right)^{\frac{1}{2}} \frac{dy}{y^{\frac{1}{2}}\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}} \ldots (22).$$

These are E, F, by (1) and (2), respectively.

$$\frac{x^{\frac{3}{2}}dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{4}}}=\frac{1}{2}\left(\frac{1+k}{2}\right)^{\frac{1}{2}}\left\{\frac{(1+ky^2)\,dy}{y^{\frac{1}{2}}\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}}-\frac{dy}{y^{\frac{1}{2}}}\right\}\ldots(23).$$

$$\frac{x^{\frac{5}{2}}dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{3}{4}}}=\frac{1}{2}\left(\frac{1+k}{2}\right)^{\frac{3}{2}}\left\{\frac{(1+ky^2)\,y^{\frac{1}{2}}dy}{\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}}-y^{\frac{1}{2}}dy\right\}..(24).$$

It is easily seen from (1), (2), that the two last are E, F; but it must be understood that they may contain circular or algebraic functions.

$$\frac{\sqrt{\{1-(1+b)x^2\}}}{\sqrt{\{1-(1-b)x^2\}}}\frac{dx}{\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}}=\frac{1+k}{2}\frac{dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{3}{4}}}\ldots(25).$$

$$\frac{\sqrt{\{1-(1-b)x^2\}}}{\sqrt{\{1-(1+b)x^2\}}}\frac{dx}{\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}}=\frac{1+k}{2}\frac{dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{4}}}\ldots(26),$$

which are E, F, by (5) and (6).

$$\frac{x^{\frac{1}{2}}dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}}=\frac{1}{2}\left(\frac{1+k}{2}\right)^{\frac{1}{2}}\left\{\frac{ky^{\frac{1}{2}}dy}{\sqrt{(1-k^2y^2)}}+\frac{y^{\frac{1}{2}}dy}{\sqrt{(1-y^2)}}\right\}\ldots(27).$$

We find

$$\frac{1}{\sqrt{(1-c^2x^2)}}=\frac{\sqrt{(1-k^2y^2)}-k\sqrt{(1-y^2)}}{1-k},\ \frac{1}{2x^2}=\frac{1+ky^2+\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}}{(1+k)^2y^2}.$$

And, by the aid of these,

$$\frac{dx}{x^{\frac{1}{2}}(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}}=\frac{(1+k)^{\frac{1}{2}}}{2^{\frac{1}{2}}(1-k)}\left\{\frac{dy}{y^{\frac{1}{2}}\sqrt{(1-y^2)}}-\frac{kdy}{y^{\frac{1}{2}}\sqrt{(1-k^2y^2)}}\right\}\ldots(28).$$

The two last are obviously E, F. And from these we easily derive the four following by means of the values of $2x^2$ and $\frac{1}{2x^2}$; but the second members, which are obviously E, F, are too complex to put down:

$$\left.\begin{array}{cc}\dfrac{x^{\frac{5}{2}}dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}} & \dfrac{x^{\frac{3}{2}}dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}}\\ \dfrac{dx}{x^{\frac{3}{2}}(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}} & \dfrac{dx}{x^{\frac{5}{2}}(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}}\end{array}\right\}\ldots\ldots(29).$$

From the transformation (c) in the former paper, we easily find

$$\frac{(1-c)^{\frac{1}{2}}(1+c)^{\frac{3}{4}}\,dx}{(1-x)^{\frac{1}{4}}(1-cx)^{\frac{1}{4}}(1+x)^{\frac{1}{2}}(1+cx)} = \frac{2^{\frac{5}{4}}dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{4}}} \;\ldots\ldots (30).$$

$$\frac{(1+c)^{\frac{5}{4}}\qquad dx}{(1-c)^{\frac{1}{2}}(1-x)^{\frac{3}{4}}(1-cx)^{\frac{3}{4}}(1+x)^{\frac{1}{2}}} = \frac{2^{\frac{3}{4}}dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{3}{4}}} \;\ldots . (31).$$

These are E, F, by (7) and (8). And, from the same source,

$$\frac{(1+cx)\,dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{2}}} = \left(\frac{1-c^2}{4}\right)^{\frac{1}{4}}\left(\frac{2}{1+c}\right)\frac{dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{2}}} \ldots (32).$$

This is an E, F by (11). By if we make $1-c^2x^2=v^2$, we see that $\dfrac{x\,dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{2}}}$ is an E, F; therefore

$$\frac{dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{2}}} \text{ is an } E, F \ldots\ldots\ldots (33).$$

The transformation (6) gives

$$\frac{dx}{x^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}(1-c^2x^2)^{\frac{1}{4}}} = \frac{-(1-c^2)^{\frac{1}{4}}\,dy}{(1-y^2)^{\frac{3}{4}}(1-c^2y^2)^{\frac{1}{2}}} \;\ldots\ldots (34),$$

$$\frac{dx}{x^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}(1-c^2x^2)^{\frac{3}{4}}} = \frac{-dy}{(1-c^2)^{\frac{1}{4}}(1-y^2)^{\frac{3}{4}}} \;\ldots\ldots\ldots (35).$$

To return now to the transformation (c),

$$\frac{\left(\frac{1-c}{2}\right)^{\frac{1}{4}}\left(\frac{1+c}{2}\right)dx}{(1+x)^{\frac{1}{4}}(1-x)(1-cx)^{\frac{1}{4}}(1+cx)} = \frac{y^{\frac{1}{2}}\,dy}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{4}}} \ldots (36),$$

an E, F by (9);

$$\frac{\left(\frac{2}{1-c}\right)^{\frac{1}{4}}\left(\frac{1+c}{2}\right)dx}{(1+x)^{\frac{3}{4}}(1-x)^{\frac{1}{2}}(1-cx)^{\frac{3}{4}}} = \frac{dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{3}{4}}} \;\ldots . (37),$$

an E, F by (35).

$$\frac{(1-c)^{\frac{1}{4}}(1+c)^{\frac{1}{2}}\,dx}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{4}}(1-cx)^{\frac{1}{4}}(1+cx)^{\frac{1}{2}}} = \frac{2^{\frac{3}{4}}dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{4}}} \ldots (38),$$

an E, F by (34).

$$\left(\frac{1-c^2}{4}\right)^{\frac{1}{4}}\left(\frac{1+c}{2}\right)\frac{dx}{(1-x^2)^{\frac{1}{4}}(1-cx)^{\frac{1}{4}}(1+cx)^{\frac{3}{4}}}=\frac{y^{\frac{1}{2}}\,dy}{(-1y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{4}}}\ \ldots(39),$$

an E, F by (27).

$$\frac{\left(\frac{2}{1-c}\right)^{\frac{1}{4}}\left(\frac{1+c}{2}\right)dx}{(1-x)^{\frac{3}{4}}(1+x)^{\frac{1}{2}}(1-cx)^{\frac{1}{2}}(1+cx)^{\frac{1}{4}}}=\frac{dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{2}}}\ \ldots(40),$$

an E, F by (33).

We now turn to the transformation (e) last paper, from which we derive

$$\frac{1+c}{\sqrt{(1-c)}}\,\frac{\sqrt{(1-cx^2)}\,dx}{x^{\frac{1}{2}}\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}}=\frac{-dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{2}}}\ \ldots.(41),$$

an E, F by (33).

$$\frac{\sqrt{(1-c^2)}\,\sqrt{(1+cx^2)}\,dx}{(1-cx^2)(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}}=\frac{-y^{\frac{1}{2}}\,dy}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{4}}}\ \ldots.\ (42),$$

an E, F by (9).

$$\frac{\sqrt{(1-c^2)}\,\sqrt{(1+cx^2)}\,dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}}=\frac{-dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{4}}}\ \ldots.\ (43),$$

an E, F by (34).

$$\frac{(1+c)^{\frac{3}{2}}}{(1-c)^{\frac{1}{2}}}\,\frac{(1-cx^2)\,dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}(1+cx^2)^{\frac{1}{2}}}=\frac{-dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{3}{4}}}\ \ldots(44),$$

an E, F by (35).

$$\frac{(1-c)^{\frac{1}{2}}(1+c)\,x^{\frac{1}{2}}\,dx}{(1-cx^2)(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}}=\frac{-y^{\frac{1}{2}}\,dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{2}}}\ \ldots.(45),$$

an E, F by (10).

$$\frac{1+c}{\sqrt{(1-c)}}\,\frac{(1-cx^2)\,dx}{x^{\frac{1}{2}}(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}}=\frac{-dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{2}}}\ \ldots(46),$$

an E, F by (11).

$$\frac{(1-c)(1+c)^{\frac{1}{2}}(1+cx^2)^{\frac{1}{2}}\,x^{\frac{1}{2}}\,dx}{(1-cx^2)\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}}=\frac{-dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{4}}}\ \ldots.\ (47),$$

an E, F by (7).

$$\frac{(1+c)^{\frac{3}{2}}}{1-c}\,\frac{(1-cx^2)\,dx}{x^{\frac{1}{2}}\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}\sqrt{(1+cx^2)}}=\frac{-dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{3}{4}}}\ldots(48),$$

an E, F by (8).

$$\frac{\sqrt{(1+c)}\,\sqrt{(1+cx^2)}\,dx}{x^{\frac{1}{2}}\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}}=\frac{-dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{4}}}\ \ldots\ldots\ (49),$$

an E, F by (6).

VI.—ON CERTAIN CONTINUED FRACTIONS.

By Percival Frost, M.A., St. John's College.

To shew that

$$e^x=\frac{1}{1-}\,\frac{x}{1+}\,\frac{x}{2-}\,\frac{x}{3+}\,\frac{2x}{4-}\,\frac{2x}{5+}\ldots\ldots\frac{rx}{2r+1+}\,\frac{(r+1)x}{2r+2-}.$$

Let $$y=e^x=\frac{1}{1+y_1},$$

$$\frac{dy_1}{dx}=-\frac{d_xy}{y^2}=-1-y_1.$$

Let $y_1=\alpha_1x$, $\alpha_1=-1$, neglecting terms in x.

Assume $$y_1=-\frac{x}{1+y_2};$$

then $$\frac{dy_1}{dx}=-1-y_1=-\frac{1}{1+y_2}+\frac{\frac{dy_2}{dx}}{(1+y_2)^2},$$

$$x\frac{dy_2}{dx}=(1+y_2)\,\{1-(1+y_2)-y_1(1+y_2)\}$$
$$=(1+y_2)\,(x-y_2).$$

If $y_2=\alpha_2x$, $\alpha_2=1-\alpha_2$, neglecting x.

Assume $$y_2=\frac{x}{2+y_3},$$

$$x\frac{dy_2}{dx}=-\frac{x}{2+y_3}-\frac{x^2\frac{dy_3}{dx}}{(2+y_3)^2};$$

then $$(2+y_3+x)\,(2x+xy_3-x)=x\,(2+y_3)-x^2\frac{dy_3}{dx},$$

$$x\frac{dy_3}{dx}=-x\,(1+y_3)-y_3\,(2+y_3)\ \ldots\ldots\ldots\ (1);$$

$y_3=\alpha_3x$ approximately

$\alpha_3=-1-2\alpha_3$, $\alpha_3=-\frac{1}{3}$.

Let $$y_3 = -\frac{x}{3+y_4},$$

$$\frac{dy_3}{dx} = -\frac{1}{3+y_4} + \frac{x\frac{dy_4}{dx}}{(3+y_4)^2}$$

$$-(1+y_3) - \frac{y_3}{x}(2+y_3) = -\frac{1}{3+y_4} + \frac{x\frac{dy_4}{dx}}{(3+y_4)^2},$$

$$x\frac{dy_4}{dx} = -(3+y_4)\{y_3(3+y_4)+2+y_4\} + 2.(3+y_4) - x$$

$$= -(3+y_4)y_4 + (2+y_4)x,$$

$$\alpha_4 = -2\alpha_4 + 2 \text{ as before, } \alpha_4 = \tfrac{3}{4}.$$

Let $$y_4 = \frac{2x}{4+y_5};$$

whence, as before, $x\frac{dy_5}{dx} = -x(2+y_5) - (4+y_5)y_5$.

According to law observed in equations for y_5, y_3.

let $$x\frac{dz}{dx} = -(r+z)x - (2r+z)z,$$

$$z = \beta x, \quad \beta = -r - 2r\beta.$$

Let $$z = -\frac{rx}{2r+1+z_1},$$

$$x\frac{dz}{dx} = -\frac{rx}{2r+1+z_1} + \frac{rx^2\frac{dz_1}{dx}}{(2r+1+z_1)^2},$$

Thus $$\frac{rx^2\frac{dz_1}{dx}}{(2r+1+z_1)^2} = -(r+z)x - (2r+1+z)z;$$

and $$rx\frac{dz_1}{dx} = -\{r(2r+1+z_1) - rx\}(2r+1+z_1)$$
$$+ r\{(2r+1)^2 + z_1(2r+1) - rx\},$$

$$x\frac{dz_1}{dx} = -(2r+1+z_1)z_1 + x(r+1+z_1),$$

$$z_1 = \beta_1 x,$$

$$\beta_1 = -(2r+1)\beta_1 + r + 1,$$

$$\beta_1 = \frac{r+1}{2r+2}.$$

Let $$z_1 = \frac{(r+1)x}{2r+2+z_2},$$

$$\frac{dz_1}{dx} = \frac{r+1}{(2r+2+z_2)} - \frac{(r+1)\,x\,\frac{dz_2}{dx}}{(2r+2+z_2)^2};$$

whence $$\frac{(r+1)\,x\,\frac{dz_2}{dx}}{2r+2+z_2} = r+1+(r+1)(2r+1+z_1),$$

$$-(r+1)(2r+2+z_2)-(r+1)\,x,$$

$$x\,\frac{dz_2}{dx} = (2r+2+z_2)(z_1-z_2-x)$$

$$= (r+1)\,x - (z_2+x)(2r+2+z_2)$$

$$= -(r+1+z_2)\,x - \{2(r+1)+z_2\}\,z_2,$$

the same form as for $x\,\dfrac{dz}{dx}$; therefore the law observed is true,

$$y_{2r} = \frac{rx}{2r+y_{2r+1}},\quad y_{2r+1} = -\frac{rx}{2r+1+y_{2r+2}},$$

and $$e^x = \frac{1}{1-}\,\frac{x}{1+}\,\frac{x}{2-}\,\frac{x}{3+}\,\frac{2x}{4-}\,\frac{2x}{5+}\,\frac{3x}{6-}\,\frac{3x}{7+}\,\ldots\ldots$$

In the same manner $l_\varepsilon(1+x)$ can be expressed in a continued fraction,
$$y = l_\varepsilon(1+x),$$

$$\frac{dy}{dx} = \frac{1}{1+x},\quad y = \frac{x}{1+y_1},$$

$$x(1+x)\,\frac{dy_1}{dx} = (1+y_1)\,x - (1+y_1)\,y_1,$$

$$\alpha_1 = \tfrac{1}{2},\quad y_1 = \frac{x}{2+y_2},$$

and $$x(1+x)\frac{dy_2}{dx} = x - (2+y_2)\,y_2,$$

$$y_2 = \frac{x}{3+y_3},\quad x(1+x)\frac{dy_3}{dx} = (4+y_3)\,x - (3+y_3)\,y_3.$$

The equation for y_{2r+1} is

$$x(1+x)\frac{dy_{2r-1}}{dx} = (r^2+y_{2r-1})\,x - y_{2r-1}(y_{2r-1}+2r-1),$$

for $$y_{2r}\,x(1+x)\,\frac{dy_{2r}}{dx} = rx - ry_{2r}(2+y_{2r});$$

and $$y_{2r-1} = \frac{rx}{2+y_{2r}},\quad y_{2r} = \frac{rx}{2r+1+y_{2r+1}},$$

the result being that

$$l_e(4x) = \frac{x}{1+}\,\frac{x}{2+}\,\frac{x}{3+}\,\frac{2x}{2+}\,\frac{2x}{5+}\,\frac{3x}{2+}\,\frac{3x}{7+}\,\frac{4x}{2+}\,\cdots\cdots$$

The method applies with great facility to many other functions.

For $\quad \tan^{-1} x = y, \quad y = \dfrac{x}{1+y_1},$

$$x(1+x^2)\frac{dy_1}{dx} = (1+y_1)x^2 - (1+y_1)y_1;$$

and, proceeding as before,

$$x(1+x^2)\frac{dy_r}{dx} = (r^2+y_r)x^2 - (2r-1+y_r)y_r,$$

and $$y_r = \frac{r^2x^2}{2r+1+y_{r+1}},$$

$$\tan^{-1}x = \frac{x}{1+}\,\frac{x^2}{3+}\,\frac{2^2x^2}{5+}\,\frac{3^2x^2}{7+}\,\frac{4^2x^2}{9+}\,\cdots.$$

For $\quad y = \tan x, \quad x\dfrac{dy_r}{dx} = -x^2 - (2r-1+y_r)y_r$

$$y_r = -\frac{x^2}{2r-1+y_{r+1}},$$

$$\tan x = \frac{x}{1-}\,\frac{x^2}{3-}\,\frac{x^2}{5-}\,\frac{x^2}{7-}\,\cdots,$$

In the case of $(1+x)^n$, we obtain by the law of formation,

$$x(1+x)\frac{dy_{2r+1}}{dx} = (n-r+ny_{2r+1})x - ry_{2r+1}(2+y_{2r+1})x$$

whence

$$x(1+x)\frac{dy_{2r+2}}{dx} = \{(r+1)(n+r+1)+(n+1)\,y_{2r+2}\}\,x$$

and $$-\,y_{2r+2}(2r+1+y_{2r+2}),$$

$$x(1+x)\frac{dy_{2r+3}}{dx} = -\,\{n-(r+1)+ny_{2r+3}\}\,x - (r+1)\,y_{2r+3}(2+y_{2r+3}),$$

which proves that the law holds generally.

Thus, $$y_{2r} = \frac{(n+r)x}{2+y_{2r+1}},$$

$$y_{2r+1} = -\frac{(n-r)x}{2r+1+y_{2r+2}},$$

$$(1+x)^n = \frac{1}{1-}\,\frac{nx}{1+}\,\frac{(n+1)x}{2-}\,\frac{(n-1)x}{3+}\,\frac{(n+2)x}{2-}\,\frac{(n-2)x}{5+}\,\frac{(n+3)x}{2-},$$

which is the required development.

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I.—ON THE PROOF OF THE PROPOSITION THAT $(Mx + Ny)^{-1}$ IS AN INTEGRATING FACTOR OF THE HOMOGENEOUS DIFFERENTIAL EQUATION $M + N\dfrac{dy}{dx} = 0$.

By G. G. STOKES, M.A. Fellow of Pembroke College.

A FALLACIOUS proof is sometimes given of this proposition, which ought to be examined. The substance of the proof is as follows.

Let us see whether it is possible to find a multiplier V, a homogeneous function of x and y, which shall render $Mdx + Ndy$ an exact differential. Let M and N be of n, and V of p dimensions; let

$$dU = V(Mdx + Ndy) \ldots\ldots\ldots (1);$$

then, on properly choosing the arbitrary constant in U, U will be a homogeneous function of $n + p + 1$ dimensions, $\Big\}$ (A),

whence, by a known theorem,

$$(n + p + 1)\, U = x\frac{dU}{dx} + y\frac{dU}{dy} = V(Mx + Ny) \ldots (2);$$

therefore, dividing (1) by (2),

$$\frac{dU}{(n + p + 1)\, U} = \frac{Mdx + Ndy}{Mx + Ny};$$

and the first side of this equation being an exact differential, it follows that the second side is so also, and consequently that $(Mx + Ny)^{-1}$ is an integrating factor.

Now the factor so found is of $-n-1$ dimensions; so that the first side of (2) is zero. In fact, we shall see that the statement (A) is not true as applied to the case in question, unless $Mx + Ny = 0$.

The general form of a function of x of n dimensions is Ax^n. The general form of a homogeneous function of x and y of n dimensions is $x^n \psi\left(\frac{y}{x}\right)$. The integral of the first is in general $\frac{Ax^{n+1}}{n+1}$, omitting the arbitrary constant; and consequently the dimensions of the function are increased by unity by integration. But in the particular case in which $n = -1$, the integral is $A \log x$, which is not a quantity of o dimensions, at least according to the definition just given, *according to which definition only* is the proposition with reference to homogeneous functions assumed in (2) true. Let us now examine in what cases U will be of $n + p + 1$ dimensions.

Putting $M = M_0 x^n$, $N = N_0 x^n$, $y = xz$, M_0 and N_0 will be functions of z alone, and we shall have

$$Mdx + Ndy = x^n \{(M_0 + N_0 z)\, dx + N_0 x\, dz\}.$$

If $M_0 + N_0 z = 0$, i.e. if $Mx + Ny = 0$, we see that x^{-n-1} will be an integrating factor. The integral, being a function of z, will be of 0 dimensions, and both sides of (2) will be zero.

If $Mx + Ny$ is not equal to 0, we may multiply and divide by $(M_0 + N_0 z)\, x$, and we have

$$Mdx + Ndy = x^{n+1} (M_0 + N_0 z) \left(\frac{dx}{x} + \frac{N_0 dz}{M_0 + N_0 z}\right).$$

Hence we see that $\{x^{n+1} (M_0 + N_0 z)\}^{-1}$ or $(Mx + Ny)^{-1}$ is an integrating factor. For this factor we have

$$U = \log (x) + \phi\left(\frac{y}{x}\right),$$

ϕ denoting the function arising from the integration with respect to z.

In this case we have $x \dfrac{dU}{dx} + y \dfrac{dU}{dy} = 1$, not $= 0$.

It may be of some interest to enquire in what cases an exact differential of any number of independent variables, in which the differential coefficients are homogeneous functions of n dimensions, has an integral which is a homogeneous function of $n + 1$ dimensions.

Let $dU = Mdx + Ndy + Pdz + \ldots$ be the exact differential. Let $y = y'x$, $z = z'x \ldots$, $M = M_0 x^n$, $N = N_0 x^n \ldots$, so that M_0, $N_0 \ldots$ are functions of y', $z' \ldots$ only; then

$$dU = x^n \{(M_0 + N_0 y' + P_0 z' \ldots)\, dx + (N_0 dy' + P_0 dz' \ldots)\, x\}.$$

First, suppose the coefficient of dx in this equation to be zero, or $Mx + Ny + Pz \ldots = 0$; then the expression for dU cannot be an exact differential unless $n = -1$. In this case U will be a function of $y', z' \ldots$, and will therefore be a homogeneous function of $n + 1$ or 0 dimensions.

Secondly, suppose the coefficient of dx not to be zero; then

$$dU = x^{n+1}(M_0 + N_0 y' \ldots)\left(\frac{dx}{x} + \frac{N_0 dy' + P_0 dz' \ldots}{M_0 + N_0 y' + P_0 z' \ldots}\right)$$

$$= (Mx + Ny + Pz \ldots)\left(\frac{dx}{x} + \frac{N_0 dy' + P_0 dz' \ldots}{M_0 + N_0 y' + P_0 z' \ldots}\right) \ldots (3).$$

Now I say that $\dfrac{N_0 dy' + P_0 dz' \ldots}{M_0 + N_0 y' + P_0 z' \ldots}$ is the exact differential of a function of the independent variables $y', z' \ldots$, or, taking $y, z \ldots$ for the independent variables instead of $y', z' \ldots$, x being supposed constant, and putting for $M_0, N_0 \ldots$ their values, that $\dfrac{Ndy + Pdz + \ldots}{Mx + Ny + Pz \ldots}$ is an exact differential.

For, putting $Mx + Ny + Pz \ldots = D$, in order that the quantity considered should be an exact differential, it is necessary and sufficient that the system of equations of which the type is $\dfrac{d\frac{N}{D}}{dz} = \dfrac{d\frac{P}{D}}{dy}$ should be satisfied. This equation gives

$$D\left(\frac{dN}{dz} - \frac{dP}{dy}\right) + P\frac{dD}{dy} - N\frac{dD}{dz} = 0.$$

Now, since $\dfrac{dN}{dz} = \dfrac{dP}{dy}$, by the conditions of $Mdx + Ndy + Pdz \ldots$ being an exact differential, the above equation becomes $P\dfrac{dD}{dy} - N\dfrac{dD}{dz} = 0$, or

$$P\left(\frac{dM}{dy}x + \frac{dN}{dy}y + \frac{dP}{dy}z \ldots\right) - N\left(\frac{dM}{dz}x + \frac{dN}{dz}y + \frac{dP}{dz}z \ldots\right) = 0.$$

Replacing $\dfrac{dM}{dy}, \dfrac{dP}{dy} \ldots$ by $\dfrac{dN}{dx}, \dfrac{dN}{dz} \ldots$ and $\dfrac{dM}{dz}, \dfrac{dN}{dz} \ldots$ by $\dfrac{dP}{dx}, \dfrac{dP}{dy} \ldots$, this equation becomes

$$P\left(\frac{dN}{dx}x + \frac{dN}{dy}y + \frac{dN}{dz}z \ldots\right) - N\left(\frac{dP}{dx}x + \frac{dP}{dy}y + \frac{dP}{dz}z \ldots\right) = 0.$$

Now
$$\frac{dN}{dx}x + \frac{dN}{dy}y + \ldots = nN,$$
$$\frac{dP}{dx}x + \frac{dP}{dy}y + \ldots = nP,$$

and therefore the above equation *is* satisfied. Hence

$$\frac{Ndy + Pdz\ldots}{Mx + Ny + Pz\ldots}, \text{ or its equal } \frac{N_0dy' + P_0dz' \ldots}{M_0 + N_0y' + P_0z' \ldots},$$

is an exact differential $d\psi(y', z' \ldots)$. Consequently equation (3) becomes

$$dU = (Mx + Ny + Pz \ldots)\, d\left\{\log x + \psi(y', z' \ldots)\right\};$$

which equation being by hypothesis integrable, it follows that

$$Mx + Ny + Pz \ldots = \phi\left\{\log x + \psi(y', z' \ldots)\right\}:$$

and $Mx + Ny \ldots$ being moreover a homogeneous function of $n+1$ dimensions, it is clear that we must have $\phi(a) = A\,e^{(n+1)a}$. Hence we have

$$dU = A\,x^{n+1}\,e^{(n+1)\psi}\,d(\log x + \psi).$$

If now $n + 1$ is not equal to 0, we have

$$U = \frac{Ax^{n+1}\,e^{(n+1)\psi}}{n+1},$$

omitting the constant; but if $n = -1$, we have

$$U = A(\log x + \psi) + C.$$

We see then that if $Mx + Ny + Pz \ldots = 0$, which can only happen when $n = -1$, U will be a homogeneous function of $n + 1$ or 0 dimensions. If $Mx + Ny + Pz \ldots\ldots$ is not equal to 0, then, if $n + 1$ is not equal to 0, and the constant in U is properly chosen, U will be a homogeneous function of $n + 1$ dimensions, but if $n + 1 = 0$, U will not be a homogeneous function of 0 dimensions, but will contain $\log x$. Of course it might equally have contained the logarithm of y or z, &c.; in fact,

$$\log x + \psi(y', z' \ldots) = \log y + \log\frac{x}{y} + \psi(y', z' \ldots)$$
$$= \log y + \chi(y', z' \ldots).$$

II.—TRANSFORMATION OF THE DIFFERENTIAL EQUATIONS OF A PLANET'S MOTION.

By B. Bronwin.

Making x, y, z the co-ordinates of a disturbed planet, x', y', z' those of the disturbing body, r and r' the radii-vectores, and

$$R = \frac{m'(xx' + yy' + zz')}{r'^3} - \frac{m'}{\{r'^2 - 2(xx' + yy' + zz') + r^2\}^{\frac{1}{2}}};$$

the known differential equations of its motion are

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{dR}{dx} = 0, \quad \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{dR}{dy} = 0, \quad \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{dR}{dz} = 0 \quad \dots (A).$$

Let i be the inclination of the plane of the orbit to the fixed plane of xy, θ the longitude of the node on the fixed plane, and ϑ its longitude on the plane of the orbit, having a fixed origin on it. Then $np = nn' \cos n'np$ (see fig. 1), or $d\vartheta = \cos i \, d\theta$.

If u and v be the co-ordinates referred to the line of the nodes, $x = u \cos\theta - v \sin\theta$, $y = u \sin\theta + v \cos\theta$. And if w be the co-ordinate on the orbit, of which v is the projection, $v = w \cos i$, $z = w \sin i$. Also, if ξ and η be the co-ordinates on the plane of the orbit, the axis of η passing through the origin of ϑ,

$$u = \xi \cos\vartheta + \eta \sin\vartheta, \quad w = -\xi \sin\vartheta + \eta \cos\vartheta.$$

From these, by eliminating u, v, and w, we have

$$x = A\xi + B\eta, \quad y = C\xi + D\eta, \quad z = E\xi + F\eta \dots (1),$$

$A = \cos\theta \cos\vartheta + \cos i \sin\theta \sin\vartheta$, $B = \cos\theta \sin\vartheta - \cos i \sin\theta \cos\vartheta$,
$C = \sin\theta \cos\vartheta - \cos i \cos\theta \sin\vartheta$, $D = \sin\theta \sin\vartheta + \cos i \cos\theta \cos\vartheta$,
$E = -\sin i \sin\vartheta$, $F = \sin i \cos\vartheta$.

Remembering that $d\vartheta = \cos i \, d\theta$, we find

$$dA = -\sin\theta \cos\theta \, d\vartheta - \cos i \cos\theta \sin\vartheta \, d\theta - \sin i \sin\theta \sin\vartheta \, di$$
$$+ \cos i \cos\theta \sin\vartheta \, d\theta + \cos^2 i \sin\theta \cos\vartheta \, d\theta$$
$$= -\sin i \sin\theta \sin\vartheta \, di - \sin^2 i \sin\theta \cos\vartheta \, d\theta;$$

and $dE = -\cos i \sin\vartheta \, di - \sin i \cos i \cos\vartheta \, d\theta$.

From these it will be easily seen that $dA = \tan i \sin\theta \, dE$.

Similar results may be obtained by differentiating the values of B, C, &c. They are, including the one just obtained,

$$\left.\begin{array}{ll} dA = \tan i \sin\theta \, dE, & dB = \tan i \sin\theta \, dF \\ dC = -\tan i \cos\theta \, dE, & dD = -\tan i \cos\theta \, dF \end{array}\right\} \dots (2).$$

As we have four quantities to determine and only three equations, we must make an assumption. Assume

$$\xi dE + \eta dF = 0,\ \ \xi dA + \eta dB = 0,\ \ \xi dC + \eta dD = 0 \ldots (3).$$

This is only equivalent to one assumption: for if we put for dA, dB, dC, dD their values from (2), we see that the two last of (3) reduce to the first. The assumption here made is the one usually made. For if ϕ = lat. v = long. on the orbit, we have $z = r\sin\phi$, $\xi = r\cos v$, $\eta = r\sin v$; and $z = E\xi + F\eta$ becomes

$$\sin\phi = E\cos v + F\sin v = \sin i\,(\sin v\cos\vartheta - \cos v\sin\vartheta)$$
$$= \sin i\sin(v - \vartheta);$$

$$\text{and}\ \ \xi dE + \eta dF = 0\ \ \text{is}\ \ \frac{d\sin\phi}{di}\,di + \frac{d\sin\phi}{d\vartheta}\,d\vartheta = 0.$$

By virtue of (3) we have

$$\frac{dx}{dt} = A\frac{d\xi}{dt} + B\frac{d\eta}{dt},\ \ \frac{dy}{dt} = C\frac{d\xi}{dt} + D\frac{d\eta}{dt},\ \ \frac{dz}{dt} = E\frac{d\xi}{dt} + F\frac{d\eta}{dt};$$

and therefore

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= A\frac{d^2\xi}{dt^2} + B\frac{d^2\eta}{dt^2} + \frac{dA}{dt}\frac{d\xi}{dt} + \frac{dB}{dt}\frac{d\eta}{dt}\\ \frac{d^2y}{dt^2} &= C\frac{d^2\xi}{dt^2} + D\frac{d^2\eta}{dt^2} + \frac{dC}{dt}\frac{d\xi}{dt} + \frac{dD}{dt}\frac{d\eta}{dt}\\ \frac{d^2z}{dt^2} &= E\frac{d^2\xi}{dt^2} + F\frac{d^2\eta}{dt^2} + \frac{dE}{dt}\frac{d\xi}{dt} + \frac{dF}{dt}\frac{d\eta}{dt}\end{aligned}\right\}\ldots(4).$$

The following equations of condition will now be necessary:

$$\left.\begin{aligned}&A^2 + C^2 + E^2 = 1,\ \ B^2 + D^2 + F^2 = 1,\ \ AB + CD + EF = 0,\\ &AdA + CdC + EdE = 0,\ \ \ BdB + DdD + FdF = 0,\\ &AdB + CdD + EdF = 0,\ \ \ BdA + DdC + FdE = 0,\end{aligned}\right\}\ldots(5).$$

These may be derived as follows. In $r^2 = x^2 + y^2 + z^2 = \xi^2 + \eta^2$ put for x, y, and z their values from (1); the coefficients of ξ^2, $\xi\eta$, η^2, equalled to zero, will give the three first. Differentiation of the first and second will give the fourth and fifth: these last, by means of (3), will give the sixth and seventh. Or if in $xdx + ydy + zdz = \xi d\xi + \eta d\eta$ for x, dx, &c. we substitute their values from (1), the coefficients of the several terms of the result equalled to zero will give them all. And they may be all verified by putting for A, B, &c. and their differentials their values.

We easily derive from (4), attending to (5),

$$\left.\begin{aligned} A\frac{d^2x}{dt^2}+C\frac{d^2y}{dt^2}+E\frac{d^2z}{dt^2}=\frac{d^2\xi}{dt^2} \\ B\frac{d^2x}{dt^2}+D\frac{d^2y}{dt^2}+F\frac{d^2z}{dt^2}=\frac{d^2\eta}{dt^2} \end{aligned}\right\}\ldots\ldots(6),$$

and from (1), having regard to (5),

$$\frac{\mu}{r^3}(Ax+Cy+Ez)=\frac{\mu\xi}{r^3},\quad \frac{\mu}{r^3}(Bx+Dy+Fz)=\frac{\mu\eta}{r^3}\ \ldots(7).$$

Multiply the first of (A) by A, the second by C, the third by E, and add the results; we have by (6) and (7),

$$\frac{d^2\xi}{dt^2}+\frac{\mu\xi}{r^3}+A\frac{dR}{dx}+C\frac{dR}{dy}+E\frac{dR}{dz}=0.$$

Again, multiply the first of (A) by B, the second by D, the third by F; add results: we have by (6) and (7),

$$\frac{d^2\eta}{dt^2}+\frac{\mu\eta}{r^3}+B\frac{dR}{dx}+D\frac{dR}{dy}+F\frac{dR}{dz}=0.$$

But

$$\frac{dR}{d\xi}=\frac{dR}{dx}\frac{dx}{d\xi}+\frac{dR}{dy}\frac{dy}{d\xi}+\frac{dR}{dz}\frac{dz}{d\xi}=A\frac{dR}{dx}+C\frac{dR}{dy}+E\frac{dR}{dz}\ \text{by (1)}.$$

In like manner $\dfrac{dR}{d\eta}=B\dfrac{dR}{dx}+D\dfrac{dR}{dy}+F\dfrac{dR}{dz}$.

These values of $\dfrac{dR}{d\xi}$, $\dfrac{dR}{d\eta}$ reduce the equations last obtained to

$$\frac{d^2\xi}{dt^2}+\frac{\mu\xi}{r^3}+\frac{dR}{d\xi}=0,\quad \frac{d^2\eta}{dt^2}+\frac{\mu\eta}{r^3}+\frac{dR}{d\eta}=0\ \ldots(B).$$

The equations (B) are the same as they would be if the plane of the orbit were fixed.

Since R is independent of the position of the fixed plane, if we make that plane coincide with the orbit, the quantity $xx'+yy'+zz'$ will become $x'\xi+y'\eta$, or $M\xi+N\eta$. And if we transform that quantity, it will become $M\xi+N\eta$, M and N having the same values in both cases, though functions of different quantities in the two cases. Consequently $\dfrac{dR}{d\xi}$, $\dfrac{dR}{d\eta}$ will have the same values, whatever be the position of the fixed plane.

Make $\frac{\xi d\eta - \eta d\xi}{dt} = h$. Then by (3),

$$\frac{dA}{dt}\,\frac{d\xi}{dt} + \frac{dB}{dt}\,\frac{d\eta}{dt} = \frac{dA}{dt}\cdot\frac{\eta d\xi - \xi d\eta}{\eta dt} = -\frac{h}{\eta}\,\frac{dA}{dt},$$

$$\frac{dC}{dt}\,\frac{d\xi}{dt} + \frac{dD}{dt}\,\frac{d\eta}{dt} = \frac{dC}{dt}\cdot\frac{\eta d\xi - \xi d\eta}{\eta dt} = -\frac{h}{\eta}\,\frac{dC}{dt},$$

$$\frac{dE}{dt}\,\frac{d\xi}{dt} + \frac{dF}{dt}\,\frac{d\eta}{dt} = \frac{dE}{dt}\cdot\frac{\eta d\xi - \xi d\eta}{\eta dt} = -\frac{h}{\eta}\,\frac{dE}{dt}.$$

Substituting these values in (4), they become

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= A\frac{d^2\xi}{dt^2} + B\frac{d^2\eta}{dt^2} - \frac{h}{\eta}\,\frac{dA}{dt}\\ \frac{d^2y}{dt^2} &= C\frac{d^2\xi}{dt^2} + D\frac{d^2\eta}{dt^2} - \frac{h}{\eta}\,\frac{dC}{dt}\\ \frac{d^2z}{dt^2} &= E\frac{d^2\xi}{dt^2} + F\frac{d^2\eta}{dt^2} - \frac{h}{\eta}\,\frac{dE}{dt}\end{aligned}\right\} \;\ldots\ldots (8).$$

These last may be changed by (3) into

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= A\frac{d^2\xi}{dt^2} + B\frac{d^2\eta}{dt^2} + \frac{h}{\xi}\,\frac{dB}{dt}\\ \frac{d^2y}{dt^2} &= C\frac{d^2\xi}{dt^2} + D\frac{d^2\eta}{dt^2} + \frac{h}{\xi}\,\frac{dD}{dt}\\ \frac{d^2z}{dt^2} &= E\frac{d^2\xi}{dt^2} + F\frac{d^2\eta}{dt^2} + \frac{h}{\xi}\,\frac{dF}{dt}\end{aligned}\right\} \;\ldots\ldots (9).$$

By (2) we find

$$\left.\begin{aligned}\left(\frac{dA}{dt}\right)^2 + \left(\frac{dC}{dt}\right)^2 + \left(\frac{dE}{dt}\right)^2 &= \frac{1}{\cos^2 i}\cdot\left(\frac{dE}{dt}\right)^2\\ \left(\frac{dB}{dt}\right)^2 + \left(\frac{dD}{dt}\right)^2 + \left(\frac{dF}{dt}\right)^2 &= \frac{1}{\cos^2 i}\cdot\left(\frac{dF}{dt}\right)^2\end{aligned}\right\} \;\ldots\ldots (10).$$

We now easily find from (8) and (9), having regard to (5) and (10),

$$\left.\begin{aligned}\frac{dA}{dt}\,\frac{d^2x}{dt^2} + \frac{dC}{dt}\,\frac{d^2y}{dt^2} + \frac{dE}{dt}\,\frac{d^2z}{dt^2} &= -\frac{h}{\eta\cos^2 i}\left(\frac{dE}{dt}\right)^2\\ \frac{dB}{dt}\,\frac{d^2x}{dt^2} + \frac{dD}{dt}\,\frac{d^2y}{dt^2} + \frac{dF}{dt}\,\frac{d^2z}{dt^2} &= \frac{h}{\xi\cos^2 i}\left(\frac{dF}{dt}\right)^2\end{aligned}\right\} \ldots (11).$$

Putting for x, y, and z their values from (1), we find in consequence of (5),

$$\left.\begin{array}{l}\frac{\mu}{r^3}\left(\frac{dA}{dt}x+\frac{dC}{dt}y+\frac{dE}{dt}z\right)=0\\ \frac{\mu}{r^3}\left(\frac{dB}{dt}x+\frac{dD}{dt}y+\frac{dF}{dt}z\right)=0\end{array}\right\} \dots (12).$$

Multiply first of (A) by $\frac{dA}{dt}$, second by $\frac{dC}{dt}$, third by $\frac{dE}{dt}$; add results: by (11) and (12) we obtain

$$-\frac{h}{\eta\cos^2 i}\left(\frac{dE}{dt}\right)^2+\frac{dA}{dt}\frac{dR}{dx}+\frac{dC}{dt}\frac{dR}{dy}+\frac{dE}{dt}\frac{dR}{dz}.$$

Again, multiply first of (A) by $\frac{dB}{dt}$, second by $\frac{dD}{dt}$, third by $\frac{dF}{dt}$; add results: by (11) and (12) we obtain

$$\frac{h}{\xi\cos^2 i}\left(\frac{dF}{dt}\right)^2+\frac{dB}{dt}\frac{dR}{dx}+\frac{dD}{dt}\frac{dR}{dy}+\frac{dF}{dt}\frac{dR}{dz}.$$

Putting for dA, dB, dC, dD their values from (2), the two last found equations reduce to

$$\left.\begin{array}{l}\frac{h}{\cos i}.\frac{dE}{dt}=\quad\eta\sin i\sin\theta\frac{dR}{dx}-\eta\sin i\cos\theta\frac{dR}{dy}+\eta\cos i\,.\frac{dR}{dz}\\ \frac{h}{\cos i}.\frac{dF}{dt}=-\xi\sin i\sin\theta\frac{dR}{dx}+\xi\sin i\cos\theta\frac{dR}{dy}-\xi\cos i\,.\frac{dR}{dz}\end{array}\right\}\dots (13),$$

Now, by (1) and (2),

$$\left.\begin{array}{l}\frac{dR}{dF}=\frac{dR}{dx}\frac{dx}{dF}+\frac{dR}{dy}\frac{dy}{dF}+\frac{dR}{dz}\frac{dz}{dF}=\eta\frac{dR}{dx}\frac{dB}{dF}+\eta\frac{dR}{dy}\frac{dD}{dF}+\eta\frac{dR}{dz}\\ \qquad=\eta\tan i\sin\theta\frac{dR}{dx}-\eta\tan i\cos\theta\frac{dR}{dy}+\eta\frac{dR}{dz}.\\ \text{In like manner}\\ \frac{dR}{dE}=\xi\tan i\sin\theta\frac{dR}{dx}-\xi\tan i\cos\theta\frac{dR}{dy}+\xi\frac{dR}{dz}.\end{array}\right\}\dots (14).$$

By means of (14), (13) will become

$$\frac{h}{\cos^2 i}\frac{dE}{dt}=\frac{dR}{dF},\quad \frac{h}{\cos^2 i}\frac{dF}{dt}=-\frac{dR}{dE}\dots (15).$$

To follow a usual notation, make

$$p=-E=\sin i\sin\vartheta,\quad q=F=\sin i$$

and (15) become

$$dp = -\frac{\cos^2 i\, dt}{h}\frac{dR}{dq}, \quad dq = \frac{\cos^2 i\, dt}{h}\frac{dR}{dp} \dots (16),$$

where $\cos^2 i = 1 - p^2 - q^2$.

From equations (16) the following formulæ, which are sometimes used, may easily be deduced:

$$d\theta = -\frac{dt}{h \sin i}\frac{dR}{di}, \quad d\vartheta = -\frac{\cos i\, dt}{h \sin i}\frac{dR}{di}, \quad di = \frac{dt}{h \sin i}\frac{dR}{d\theta}.$$

The equation $\sin\phi = E\cos v + F\sin v = \sin i \sin(v - \vartheta)$ will give the latitude by means of E and F, and we may form a differential equation for the purpose if we please. In which case we may find the variations of E and F, or of i and ϑ, due to the several powers of the disturbing force, from those of ϕ and v, in the manner pointed out in my paper on the Differential Equations of the Moon's motion in the *London, Edinburgh, and Dublin Philosophical Magazine*, Feb. 1844. Those equations are founded on the previous knowledge of equations (B) of this paper; which last are convenient for finding the co-ordinates r and v on the plane of the orbit instead of their projected values on the fixed plane. And this method is mostly the easiest way of solving the problem.

III.—ON A NEW METHOD OF OBTAINING THE EXPRESSION FOR THE SINE AND COSINE OF THE MULTIPLE ARC IN TERMS OF THE LIKE FUNCTIONS OF THE SIMPLE ARC.

By R. Moon, M.A. Fellow of Queens' College.

The subject which I propose to discuss in the following short paper is somewhat trite; but as the method I purpose to adopt possesses great simplicity, and as the artifice adopted in it may be usefully extended to other cases, I perhaps may be excused in my endeavour to call attention to it.

By De Moivre's Theorem,

$$\cos n\theta + \sqrt{(-1)}\sin n\theta = \{\cos\theta + \sqrt{(-1)}\sin\theta\}^n$$

$$= \cos^n\theta + C_1\sqrt{(-1)}\sin\theta\cos^{n-1}\theta - C_2\sin^2\theta\cos^{n-2}\theta$$

$$- C_3\sqrt{(-1)}\sin^3\theta\cos^{n-3}\theta + \&c.$$

where $C_1, C_2, C_3 \dots$ are the successive coefficients in the expansion of $(1+x)^n$;

Hence $\cos n\theta = \cos^n\theta - C_2 \sin^2\theta \cos^{n-2}\theta + C_4 \sin^4\theta \cos^{n-4}\theta - \&c.$

It will be observed, that the coefficient of $\cos^{n-2r}\theta$ in the above expression is a homogeneous function of one dimension of the quantities $C_2, C_3, C_4 \ldots$; and this function will be identical with the coefficient of $\cos^{n-2r}\theta$ in the expression

$$u = \cos^n\theta - A(1-\cos^2\theta)\cos^{n-2}\theta + A^2(1-\cos^2\theta)^2\cos^{n-4}\theta - \&c.$$

if we change A into C_2, A^2 into C_4, A^3 into C_6, &c. But

$$u = \cos^n\theta \left\{1 - A\frac{1-\cos^2\theta}{\cos^2\theta} + A^2\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)^2 - A^3\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)^3 + \&c.\right\}$$

$$= \cos^n\theta f\left\{-\left(A\frac{1-\cos^2\theta}{\cos^2\theta}\right)\right\}$$

$$= \cos^n\theta f\left\{A - \frac{A}{\cos^2\theta}\right\}$$

$$= \cos^n\theta \left\{ fA - \frac{1}{1}\frac{A}{\cos^2\theta}\frac{dfA}{dA} + \frac{1}{1.2}\left(\frac{A}{\cos^2\theta}\right)^2\frac{d^2fA}{dA^2} - \frac{1}{1.2.3}\left(\frac{A}{\cos^2\theta}\right)^3\frac{d^3fA}{dA^3} + \&c.\right.$$

and fA is the coefficient of $\cos^n\theta$ in the equivalent for u; therefore

$$fA = 1 + A + A^2 + A^3 + A^4 + \&c.$$

$$\frac{dfA}{dA^2} = 1 + 2A + 3A^2 + 4A^3 + \&c.$$

$$\frac{d^2fA}{dA^2} = 2.1 + 3.2A + 4.3A^2 + \&c.$$

$$\frac{d^3fA}{d\theta^3} = 3.2.1 + 4.3.2A + \&c.$$

$$\ldots\ldots = \ldots\ldots$$

Hence
$$u = \cos^n\theta(1 + A + A^2 + A^3 + \&c.) - \frac{\cos^{n-2}\theta}{1}(A + 2A^2 + 3A^3 + 4A^4 + \&c.) + \frac{\cos^{n-4}\theta}{1.2}(1.2A^2 + 2.3A^3 + 3.4A^4 + \&c.) - \frac{\cos^{n-6}\theta}{1.2.3}(1.2.3A^3 + 2.3.4A^4 + 3.4.5A^5 + \&c.) + \&c.;$$

and substituting for A, A^2, A^3 .... &c., we get

$$\begin{aligned}\cos n\theta = \; & \cos^n \theta\,(1 + C_2 + C_4 + C_6 + \&c.) \\ & - \frac{\cos^{n-2}\theta}{1}(C_2 + 2C_4 + 3C_6 + \&c.) \\ & + \frac{\cos^{n-4}\theta}{1.2}(1.2C_4 + 2.3C_6 + 3.4C_8 + \&c.) \\ & - \frac{\cos^{n-6}\theta}{1.2.3}(1.2.3C_6 + 2.3.4C_8 + 3.4.5C_{10} + \&c. \\ & + \&c.\end{aligned}$$

a result which may be readily verified.

Similarly we have

$$\begin{aligned}\sin n\theta &= C_1 \sin\theta \cos^{n-1}\theta - C_3 \sin^3\theta \cos^{n-3}\theta + C_5 \sin^5\theta \cos^{n-5}\theta - \&c. \\ &= \sin\theta\,\{C_1 \cos^{n-1}\theta - C_3(1-\cos^2\theta)\cos^{n-3}\theta \\ &\qquad + C_5(1-\cos^2\theta)^2 \cos^{n-5}\theta - \&c. \\ &= \sin\theta \,.\, u.\end{aligned}$$

Now the coefficient of $(\cos\theta)^{n-p}$ in u may be found from that of $(\cos\theta)^{n-p}$ in the expression

$$v = A\cos^{n-1}\theta - A^2(1-\cos^2\theta)\cos^{n-3}\theta + A^3(1-\cos^2\theta)^2\cos^{n-5}\theta - \&c.$$

by substituting C_2 for A, C_3 for A^2, C_5 for A^3, &c.

$$\text{But } v = A\cos^{n-1}\theta\left\{1 - A\,\frac{1-\cos^2\theta}{\cos^2\theta} + A^2\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)^2 - \&c.\right\}$$

$$= A\cos^{n-1}\theta\, f\left(A - \frac{A}{\cos^2\theta}\right);$$

whence we easily find

$$\sin n\theta = \sin\theta\left\{\begin{aligned} & \cos^{n-1}\theta\,(C_1 + C_3 + C_5 + \&c.) \\ & - \frac{\cos^{n-3}\theta}{1}(C_3 + 2C_5 + 3C_7 + \&c.) \\ & + \frac{\cos^{n-5}\theta}{1.2}(1.2C_5 + 2.3C_7 + 3.4C_9 + \&c.) \\ & - \frac{\cos^{n-7}\theta}{1.2.3}(1.2.3C_7 + 2.3.4C_9 + \&c.) + \&c.\end{aligned}\right.$$

We might in the same manner easily obtain the expression for $\sin n\theta$ and $\cos n\theta$ in terms of the powers of $\sin\theta$.

Liverpool, Dec. 20, 1844.

IV.—ON INDETERMINATE MAXIMA AND MINIMA.

By W. Walton, M.A., Trinity College.

Suppose that we have n equations connecting a quantity r with n quantities $x, y, z, \ldots$ and let it be proposed to determine a relation among certain parameters of these equations, such that for all values of $x, y, z, \ldots$ the value of r shall remain invariable. Problems coming under this general head may generally be solved with much elegance by the following method. Proceed by the ordinary rules to find the values of $x, y, z, \ldots$ which correspond to a maximum or minimum value of r: the result of the investigation will be an equation which, together with the n original equations, will generally serve to determine $x, y, z, \ldots$ as well as r. Having obtained these equations, we must assign such relations among the proper parameters of the equations as shall render the values of $x, y, z, \ldots$ all or some of them, according to the case, indeterminate. These relations will constitute the solution of the problem proposed. I will subjoin the solutions of several problems of the above class, which will sufficiently elucidate the principles of the method.

(1) To investigate the positions of the circular sections of the surface $$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2 \ldots\ldots(1),$$
made by planes passing through the centre of the surface.

Let (l, m, n) be the direction-cosines of any plane section through the origin, r any radius of the section, and α, β, γ, its direction-cosines. Then
$$l\alpha + m\beta + n\gamma = 0 \ldots\ldots(2),$$
and
$$\alpha^2 + \beta^2 + \gamma^2 = 1 \ldots\ldots(3);$$
also, from (1), we have
$$r^2 = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 \ldots\ldots(4).$$
Since r must be constant for a circular section, differentiating (2), (3), (4), on this hypothesis with respect to α, β, γ, we have
$$ld\alpha + md\beta + nd\gamma = 0,$$
$$\alpha d\alpha + \beta d\beta + \gamma d\gamma = 0,$$
$$a^2\alpha d\alpha + b^2\beta d\beta + c^2\gamma d\gamma = 0.$$
Multiplying these three equations in order by $1, -\lambda\mu, -\lambda$, and adding, we get, $\lambda\mu$ and λ being arbitrary multipliers,
$$\left.\begin{aligned} l &= \lambda(\mu - a^2)\alpha \\ m &= \lambda(\mu - b^2)\beta \\ n &= \lambda(\mu - c^2)\gamma \end{aligned}\right\} \ldots\ldots(5).$$

From (2) and (5) there is

$$\frac{l^2}{\mu - a^2} + \frac{m^2}{\mu - b^2} + \frac{n^2}{\mu - c^2} = 0 \ldots\ldots\ldots (6):$$

from (3) and (5),

$$\frac{l^2}{(\mu - a^2)^2} + \frac{m^2}{(\mu - b^2)^2} + \frac{n^2}{(\mu - c^2)^2} = \lambda^2 \ldots\ldots (7).$$

From (3) and (4),

$$\lambda^2 r^2 = \frac{l^2 a^2}{(\mu - a^2)^2} + \frac{m^2 b^2}{(\mu - b^2)^2} + \frac{n^2 c^2}{(\mu - c^2)^2}$$

$$= \mu\lambda^2, \quad \text{by } \mu(7) - (6),$$

whence $$r^2 = \mu \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Now, the section being circular, α, β, γ, must each be an indeterminate quantity: hence, as is evident from (5), the value of λ in (7) must be of the form $\frac{0}{0}$. We must have, therefore, one of the three relations

$$\left\{\begin{matrix} l^2 = 0 \\ r^2 = a^2 \end{matrix}\right\}, \quad \left\{\begin{matrix} m^2 = 0 \\ r^2 = b^2 \end{matrix}\right\}, \quad \left\{\begin{matrix} n^2 = 0 \\ r^2 = c^2 \end{matrix}\right\}.$$

Corresponding to each of these relations, we must have respectively, from (2), (5), (8),

$$\frac{m^2}{a^2 - b^2} + \frac{n^2}{a^2 - c^2} = 0, \quad \frac{n^2}{b^2 - c^2} + \frac{l^2}{b^2 - a^2} = 0, \quad \frac{l^2}{c^2 - a^2} + \frac{m^2}{c^2 - b^2} = 0:$$

of which relations the second alone is possible. Thus we see that there are two circular sections given by the equation

$$\frac{l}{n} = \pm\left(\frac{a^2 - c^2}{b^2 - c^2}\right)^{\frac{1}{2}}, \quad m = 0;$$

the radii of both being equal to b.

If any other possible relation were established between l, m, n, the values of α, β, γ, r, might be found from the equations (5), (6), (7), (8); so that r would be ascertained both in magnitude and position. Thus the problem with which we have been occupied is a nugatory case of a general problem of maximum or minimum values.

(2) To investigate the positions of the circular sections of an ellipsoid made by planes passing through the centre.

The equation to the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

whence $$\frac{1}{r^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2},$$

it is evident that all the equations of the last problem will be adapted to the case of the ellipsoid by writing $\frac{1}{r^2}, \frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$, respectively in place of r^2, a^2, b^2, c^2.

Thus we see that the positions of the circular sections will be determined by one of the relations

$$\frac{m^2}{\frac{1}{a^2}-\frac{1}{b^2}} + \frac{n^2}{\frac{1}{a^2}-\frac{1}{c^2}} = 0, \quad \frac{n^2}{\frac{1}{b^2}-\frac{1}{c^2}} + \frac{l^2}{\frac{1}{b^2}-\frac{1}{a^2}} = 0, \quad \frac{l^2}{\frac{1}{c^2}-\frac{1}{a^2}} + \frac{m^2}{\frac{1}{c^2}-\frac{1}{b^2}} = 0,$$

the second only of which is possible. Thus we see that two circular sections are determined by the equations

$$m = 0, \quad \frac{l}{n} = \pm \left\{\frac{\frac{1}{b^2}-\frac{1}{a^2}}{\frac{1}{c^2}-\frac{1}{b^2}}\right\}^{\frac{1}{2}}.$$

(3) To find the position of a plane passing through the centre of an ellipsoid, such that the perpendiculars from the centre upon the tangent planes at every point of the curve of section shall be equal.

The equation to the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \,\ldots\ldots\,(1),$$

we shall have, p being one of the perpendiculars,

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \,\ldots\ldots\,(2).$$

Let the equation to the plane of section be

$$lx + my + nz = 0 \,\ldots\ldots\,(3).$$

Since p is to be constant, we have from (3), (1), (2),

$$l\,dx + m\,dy + n\,dz = 0,$$

$$\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy + \frac{z}{c^2}\,dz = 0,$$

$$\frac{x}{a^4}\,dx + \frac{y}{b^4}\,dy + \frac{z}{c^4}\,dz = 0.$$

Multiplying these equations by $1, -\lambda\mu, -\lambda$, and adding, we shall get, $\lambda\mu$ and λ being arbitrary,

$$\left.\begin{aligned} l &= \frac{\lambda}{a^2}\left(\mu - \frac{1}{a^2}\right)x, \\ m &= \frac{\lambda}{b^2}\left(\mu - \frac{1}{b^2}\right)y, \\ n &= \frac{\lambda}{c^2}\left(\mu - \frac{1}{c^2}\right)z. \end{aligned}\right\} \quad \ldots\ldots (4).$$

From (1) and (4),

$$\frac{l^2a^2}{\left(\mu - \frac{1}{a^2}\right)^2} + \frac{m^2b^2}{\left(\mu - \frac{1}{b^2}\right)^2} + \frac{n^2c^2}{\left(\mu - \frac{1}{c^2}\right)^2} = \lambda^2 \ldots . (5);$$

from (3) and (4),

$$\frac{l^2a^2}{\mu - \frac{1}{a^2}} + \frac{m^2b^2}{\mu - \frac{1}{b^2}} + \frac{n^2c^2}{\mu - \frac{1}{c^2}} = 0 \quad \ldots\ldots\ldots (6).$$

From (2) and (4),

$$\frac{\lambda^2}{p^2} = \frac{l^2}{\left(\mu - \frac{1}{a^2}\right)^2} + \frac{m^2}{\left(\mu - \frac{1}{b^2}\right)^2} + \frac{n^2}{\left(\mu - \frac{1}{c^2}\right)^2}$$

$$= \mu\lambda^2, \text{ by } \mu(5) - (6),$$

whence

$$\frac{1}{p^2} = \mu \ldots\ldots\ldots\ldots\ldots (7).$$

Since λ, given by (5), must be indeterminate, in order to render x, y, z, indeterminate in (4), we must have one of the relations

$$\left\{\begin{aligned} l &= 0 \\ \mu &= \frac{1}{a^2} \end{aligned}\right\}, \quad \left\{\begin{aligned} m &= 0 \\ \mu &= \frac{1}{b^2} \end{aligned}\right\}, \quad \left\{\begin{aligned} n &= 0 \\ \mu &= \frac{1}{c^2} \end{aligned}\right\}.$$

Corresponding respectively to these relations we have, from (3), (4), (7),

$$\frac{m^2b^2}{\frac{1}{a^2} - \frac{1}{b^2}} + \frac{n^2c^2}{\frac{1}{a^2} - \frac{1}{c^2}} = 0, \qquad \frac{n^2c^2}{\frac{1}{b^2} - \frac{1}{c^2}} + \frac{l^2a^2}{\frac{1}{b^2} - \frac{1}{a^2}} = 0,$$

$$\frac{l^2u^2}{\frac{1}{c^2} - \frac{1}{a^2}} + \frac{m^2b^2}{\frac{1}{c^2} - \frac{1}{b^2}} = 0,$$

of which the second alone is possible.

Thus we see that there are two sections determined by the equations

$$m = 0, \quad \frac{l}{n} = \pm \frac{c}{a} \left\{ \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{c^2} - \frac{1}{b^2}} \right\}^{\frac{1}{2}},$$

the constant magnitude of the perpendicular being $= b^2$.

V.—ON THE INVERSE ELLIPTIC FUNCTIONS.

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THE properties of the inverse elliptic functions have been the object of the researches of the two illustrious analysts, Abel and Jacobi. Among their most remarkable ones may be reckoned the formulæ given by Abel (*Œuvres*, t. I. p. 213), in which the functions ϕa, fa, Fa, (corresponding to Jacobi's $\sin am.a$, $\cos am.a$, $\Delta am.a$, though not precisely equivalent to these, Abel's radical being $[(1 - c^2x^2)(1 + e^2x^2)]^{\frac{1}{2}}$, and Jacobi's, like that of Legendre's $[(1 - x^2)(1 - k^2x^2)]^{\frac{1}{2}}$), are expressed in the form of fractions, having a common denominator; and this, together with the three numerators, resolved into a doubly infinite series of factors; *i.e.* the general factor contains two independent integers. These formulæ may conveniently be referred to as "Abel's double factorial expressions" for the functions ϕ, f, F. By dividing each of these products into an infinite number of partial products, and expressing these by means of circular or exponential functions, Abel has obtained (p. 216–218) two other systems of formulæ for the same quantities, which may be referred to as "Abel's first and second single factorial systems." The theory of the functions forming the above numerators and denominator, is mentioned by Abel in a letter to Legendre (*Œuvres*, t. II. p. 259), as a subject to which his attention had been directed, but none of his researches upon them have ever been published. Abel's double factorial expressions have nowhere any thing analogous to them in Jacobi's *Fund. Nova;* but the system of formulæ analogous to the first single factorial system is given by Jacobi (p. 86), and the second system is implicitly contained in some of the subsequent formulæ. The functions forming the numerator and denominator of $\sin am.u$, Jacobi represents, omitting a constant factor, by $H, (u)$, $\Theta(u)$; and proceeds to investigate the properties of these new functions. This he principally effects by means of a very remarkable equation of the form

$$l\Theta(u) = \tfrac{1}{2}Au^2 + B\int_0 du.\int_0 du \sin^2 amu,$$

(*Fund. Nova*, p. 145, 133), by which $\Theta(u)$ is made to depend on the known function sin *am.u.* The other two numerators are easily expressed by means of the two functions H, Θ.

From the omission of Abel's double factorial expressions, which are the only ones which display clearly the real nature of the functions in the numerators and denominators; and besides, from the different form of Jacobi's radical, which complicates the transformation from an impossible to a possible argument, it is difficult to trace the connection between Jacobi's formulæ; and in particular to account for the appearance of an exponential factor which runs through them. It would seem therefore natural to make the whole theory depend upon the definitions of the new transcendental functions to which Abel's double factorial expressions lead one, even if these definitions were not of such a nature, that one only wonders they should never have been assumed *à priori* from the analogy of the circular functions sin. cos. and quite independently of the theory of elliptic integrals. This is accordingly what I have done in the present paper, in which therefore I assume no single property of elliptic functions, but demonstrate them all, from my fundamental equations. For the sake however of comparison, I retain entirely the notation of Abel. Several of the formulæ that will be obtained are new.

The infinite product

$$x\Pi\left(1+\frac{x}{m\omega}\right)\ldots\ldots\ldots\ldots\ldots(1),$$

where m receives the integer values $\pm 1, \pm 2, \ldots \pm r$, converges, as is well known, as r becomes indefinitely great to a determinate function $\sin\frac{\pi x}{\omega}$ of x; the theory of which might, if necessary, be investigated from this property assumed as a definition. We are thus naturally led to investigate the properties of the new transcendant

$$u = x\Pi\Pi\left(1+\frac{x}{m\omega+n\nu i}\right)\ldots\ldots\ldots(2):$$

m and n are integer numbers, positive or negative; and it is supposed that whatever positive value is attributed to either of these, the corresponding negative one is also given to it. $i=\sqrt{(-1)}$, ω and ν are real positive quantities. (At least this is the standard case, and the only one we shall explicitly consider. Many of the formulæ obtained are true, with slight modifications, whatever ω and ν represent, provided only $\omega : \nu i$

be not a real quantity; for if it were so, $m\omega + nvi$ for some values of m, n would vanish, or at least become indefinitely small, and u would cease to be a determinate function of x).*

Now the value of the above expression, or, as for the sake of shortness it may be written, of the function

$$u = x\Pi\Pi \left\{1 + \frac{x}{(m, n)}\right\} \dots\dots\dots\dots (3),$$

depends in a remarkable manner on the mode in which the superior limits of m, n are assigned. Imagine m, n to have any positive or negative integer values satisfying the equation

$$\phi(m^2, n^2) < T.$$

Consider, for greater distinctness, m, n as the co-ordinates of a point; the equation $\phi(m^2, n^2) = T$ belongs to a certain curve symmetrical with respect to the two axes. I suppose besides that this is a continuous curve without multiple points, and such that the minimum value of a radius vector through the origin continually increases as T increases, and becomes infinite with T. The curve may be *analytically* discontinuous, this is of no importance. The condition with respect to the limits is then that m and n must be integer values denoting the co-ordinates of a point *within* the above curve, the whole system of such integer values being successively taken for these quantities.

Suppose, next, u' denotes the same function as u, except that the limiting condition is

$$\phi'(m^2, n^2) < T' \dots\dots\dots\dots (5).$$

The curve $\phi'(m^2, n^2) = T'$ is supposed to possess the same properties with the other limiting curve, and, for greater distinctness, to lie entirely outside of it; but this last condition is nonessential.

These conditions being satisfied, the ratio $u' : u$ is very easily determined in the limiting case of T and T' infinite. In fact

$$\frac{u'}{u} = \Pi\Pi \left\{1 + \frac{x}{(m, n)}\right\} \dots\dots\dots\dots (6),$$

or

$$\frac{lu'}{u} = \Sigma\Sigma l \left\{1 + \frac{x}{(m, n)}\right\} \dots\dots\dots\dots (7),$$

the limiting conditions being

$$\phi(m^2, n^2) > T \dots\dots\dots\dots (8).$$
$$\phi'(m^2, n^2) < T'.$$

* I have examined the [illegible] of impossible values of ω and v in a paper which I am prep[illegible]or *Crelles* [illegible]al.

Now $$l\left\{1+\frac{x}{(m,n)}\right\}=\frac{x}{(m,n)}-\tfrac{1}{2}\cdot\frac{x^2}{(m,n)^2}+\ldots\ldots(9),$$

$$\frac{lu'}{u}=x.\Sigma\Sigma\frac{1}{(m,n)}-\tfrac{1}{2}x^2.\Sigma\Sigma\frac{1}{(m,n)^2}+\ldots.(10).$$

Or, the alternate terms vanishing on account of the positive and negative values destroying each other,

$$l\frac{u'}{u}=-\tfrac{1}{2}x^2.\Sigma\Sigma\frac{1}{(m,n)^2}-\tfrac{1}{4}x^4.\Sigma\Sigma\frac{1}{(m,n)^4}-\ldots.(11).$$

In general $\Sigma\Sigma\psi(m,n)=\int\int\psi(m,n)\,dmdn+P\ldots.(12)$,
P denoting a series the first term of which is of the form $C\psi(m,n)$, and the remaining ones depending on the differential coefficients of this quantity with respect to m and n. The limits between which the two sides are to be taken, are identical.

In the present case, supposing T and T' indefinitely great, it is easy to see that the first term of the expression for $l\dfrac{u'}{u}$ is the only one which is not indefinitely small; and we have

$$l\frac{u'}{u}=-\tfrac{1}{2}Ax^2,\quad\text{or}\quad u'=u\varepsilon^{-\frac{1}{2}Ax^2}\ldots\ldots.(13),$$

where $$A=\int\int\frac{dm.dn}{(m,n)^2}=\int\int\frac{dm.dn}{(m\omega+nvi)^2}\ \ldots\ldots.(14);$$

the limits of the integration being given by

$$\phi(m^2,n^2)>T\ldots\ldots\ldots\ldots.(15).$$
$$\phi'(m^2,n^2)<T'.$$

Some particular cases are important. Suppose the limits of u' are given by $m^2\omega^2<T^2,\ n^2v^2<T^2\ \ldots\ldots\ldots.(16)$.

And thus of u, by $m^2\omega^2+n^2v^2.<T^2\ \ldots\ldots\ldots\ldots(17)$,

we have $$A=\int\int\frac{dmdn}{(m\omega+nvi)^2}\ \ldots\ldots\ldots\ (18),$$

$$=-\frac{1}{\omega}\int dn.\left\{\frac{1}{T+nvi}-\frac{1}{\surd(T^2-n^2v^2)+nvi}-\frac{1}{-\surd(T^2-n^2v^2)+nvi}-\frac{1}{-T+nvi}\right\}$$

$$=-\frac{2}{\omega}\int dn.\left\{\frac{T}{T^2+n^2v^2}-\frac{\surd(T^2-n^2v^2)}{T^2}\right\}\quad(nv=-T,\ nv=T)$$

$$=-\frac{2}{\omega v}\int_{-1}^{1}d\theta.\left\{\frac{1}{1+\theta^2}-\surd(1-\theta^2.)\right\}=-\frac{2}{\omega v}(\pi-\pi)=0.$$

Or, in this case, $$u' = u \quad\ldots\ldots\ldots\ldots\ldots\ldots (19).$$

Again, let the limits of u' be

$$m^2\omega^2 < R'^2, \quad n^2v^2 < S'^2 \quad\ldots\ldots\ldots\ldots (20),$$

and those of u, $$m^2\omega^2 < R^2, \quad n^2v^2 < S^2 \quad\ldots\ldots\ldots\ldots (21).$$

$$A = \iint \frac{dmdn}{(m\omega + nvi)^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

$$= -\frac{1}{\omega}\int dn \cdot \left\{\frac{1}{R' + nvi} - \frac{1}{R + nvi} + \frac{1}{-R + nvi} - \frac{1}{-R' + nvi}\right\},$$

where the limits are $n^2v^2 < S'^2$, for the terms containing R', $n^2v^2 < S^2$. for the terms containing R

$$= -\frac{2}{\omega vi} \cdot l\,\frac{R' + S'i}{R' - S'i} \cdot \frac{R - Si}{R + Si} \quad\ldots\ldots\ldots\ldots\ldots\ldots (23),$$

$$= -\frac{4}{\omega v}(\lambda' - \lambda), \quad \text{if } \lambda' = \tan^{-1}\frac{S'}{R'}, \; \lambda = \tan^{-1}\frac{S}{R},$$

the arcs λ, λ' being included between the limits $0, \frac{\pi}{2}$. Hence

$$u' = \varepsilon^{2(\lambda' - \lambda)\frac{x^2}{\omega v}}\, u \quad\ldots\ldots\ldots\ldots\ldots (24).$$

In particular if $\frac{S'}{R'} = \frac{S}{R}$, $u' = u$. If $\frac{S'}{R'} = 0$, $\frac{S}{R} = 1$, $u' = u\varepsilon^{-\frac{1}{2}\beta x^2}$ if $\frac{S'}{R'} = \infty$, $\frac{S}{R} = 1$, $u' = u\varepsilon^{\frac{1}{2}\beta x^2}$ where $\beta = \frac{\pi}{\omega v}$, for which quantity it will continue to be used.

We may now completely define the functions whose properties are to be investigated. Writing, for shortness,

$$(m, n) = m\omega + nvi \quad\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

$$(\overline{m}, n) = (m + \tfrac{1}{2})\,\omega + nvi.$$

$$(m, \overline{n}) = m\omega + (n + \tfrac{1}{2})\,vi.$$

$$(\overline{m}, \overline{n}) = (m + \tfrac{1}{2})\,\omega + (n + \tfrac{1}{2})\,vi.$$

We may put $$\gamma x = x\Pi\Pi\left\{1 + \frac{x}{(m, n)}\right\} \quad\ldots\ldots\ldots\ldots (B).$$

$$gx = \Pi\Pi\left\{1 + \frac{x}{(\overline{m}, n)}\right\}.$$

$$Gx = \Pi\Pi\left\{1 + \frac{x}{(m, \overline{n})}\right\}.$$

$$\mathfrak{G}x = \Pi\Pi\left\{1 + \frac{x}{(\overline{m}, \overline{n})}\right\}.$$

The limits being given respectively by the equations $\bmod^s(m,n)< T$, $\bmod^s(\overline{m},n)< T$, $\bmod^s(m,\overline{n})< T$, $\bmod^s(\overline{m},\overline{n})< T$, T being finally infinite. The system of values $m=0$, $n=0$, is of course omitted in γx.

The functions γx, gx, Gx, $\mathfrak{G}x$, are all of them real finite functions of x, possessing properties analogous to that of (u). Thus, representing any one of them by Jx, we have

$$Jx = \varepsilon^{\mp\frac{1}{2}\beta x^2} J_{\pm\beta} x \ldots\ldots\ldots\ldots\ldots (C),$$

where $J_{\pm\beta}x$ is the same as Jx, only for $J_\beta x$ the limits are given by $m^2\omega^2$ or $(m+\frac{1}{2})^2\omega^2 < R^2$, $n^2 v^2$ or $(n+\frac{1}{2})^2 v^2 < S$, ($R$, S, and $\frac{R}{S}$ infinite), and for $J_{-\beta}x$, by the same formulæ, (R, S, and $\frac{S}{R}$ infinite). It is to this equation that the most characteristic properties of the functions Jx are due.

The following equations are deduced immediately from the above definitions:

$$\gamma(-x) = -\gamma x,\ g(-x) = gx,\ G(-x) = Gx,\ \mathfrak{G}(-x) = \mathfrak{G}x \ldots (D),$$

$$\gamma(0) = 0,\ g(0) = 1,\ G(0) = 1,\ \mathfrak{G}0 = 1 \ldots\ldots\ldots\ldots\ldots (E),$$

$$\gamma' 0 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (F).$$

Suppose $\gamma_1 x$, $g_1 x$, $G_1 x$, $\mathfrak{G}_1 x$, are the values that would have been obtained for γx, gx, Gx, $\mathfrak{G}x$; by interchanging ω and v,—then changing x into xi, and interchanging m and n, by which means the limiting equations are the same in the two cases, we obtain the following system of equations :

$$\gamma_1(xi) = i\gamma x \ldots\ldots\ldots\ldots\ldots\ldots (G).$$
$$g_1(xi) = Gx.$$
$$G_1(xi) = gx.$$
$$\mathfrak{G}_1(xi) = \mathfrak{G}x.$$

Or otherwise,

$$\gamma(xi) = i\gamma_1 x \ldots\ldots\ldots\ldots\ldots (H).$$
$$g(xi) = G_1 x.$$
$$G(xi) = g_1 x.$$
$$\mathfrak{G}(xi) = \mathfrak{G}_1 x,$$

equations which are useful in transforming almost any other property of the functions J.

The functions $J^\beta x$ are changed one into another, except as regards a constant multiplier, by the change of x into $x+\frac{\omega}{2}$.

This will be shown in a note, or it may be seen from some formulæ deduced immediately from the definitions of the functions $J_\beta x$, which will be given in the sequel.* Observing the relation between Jx and $J_\beta x$, we have in particular

$$\gamma\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{2}\beta\omega x}\,Agx\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.(G).$$

$$g\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{2}\omega\beta x}\,B\gamma x.$$

$$G\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{2}\beta\omega x}\,C\mathfrak{G}x.$$

$$\mathfrak{G}\left(x+\frac{\omega}{2}\right)=\varepsilon^{\frac{1}{2}\beta\omega x}\,DGx,$$

where A, B, C, D, are most simply determined by writing $x=0,\ x=-\frac{\omega}{2}$. Putting at the same time $\varepsilon^{\beta\omega^2}=\varepsilon^{\frac{\pi\omega}{\nu}}=q_1^{-1}$,

$$A=\gamma\,\frac{\omega}{2}\,.\,(H).$$

$$B=-\,q_1^{-\frac{1}{4}}\div\gamma\left(\frac{\omega}{2}\right).$$

$$C=G\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\div\mathfrak{G}\,\frac{\omega}{2}.$$

$$D=\mathfrak{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\div G\,\frac{\omega}{2};$$

whence also

$$G\left(\frac{\omega}{2}\right)\mathfrak{G}\left(\frac{\omega}{2}\right)=q_1^{-\frac{1}{4}}\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,(25).$$

Similarly, the functions $J_{-\beta}x$ are changed one into the other by the change of x into $x+\frac{1}{2}\nu i$. We have in the same way

$$\gamma\left(x+\frac{\nu i}{2}\right)=\varepsilon^{-\frac{1}{2}\beta\nu xi}\,A'Gx\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,(I).$$

$$g\left(x+\frac{\nu i}{2}\right)=\varepsilon^{-\frac{1}{2}\beta\nu xi}\,B'\mathfrak{G}x.$$

$$G\left(x+\frac{\nu i}{2}\right)=\varepsilon^{-\frac{1}{2}\beta\nu xi}\,C'\gamma x.$$

$$\mathfrak{G}\left(x+\frac{\nu i}{2}\right)=\varepsilon^{-\frac{1}{2}\beta\nu xi}\,D'gx.$$

* Not given in the present paper.

Whence
$$A' = \gamma\,\frac{\upsilon i}{2} \dots\dots\dots\dots\dots\dots (J).$$
$$B' = g\,\frac{\upsilon i}{2} = q^{-\frac{1}{4}} \div \mathfrak{G}\,\frac{\upsilon i}{2},$$
$$C' = -\,q^{-\frac{1}{4}} \div \gamma\,\frac{\upsilon i}{2},$$
$$D' = \mathfrak{G}\,\frac{\upsilon i}{2} = q^{-\frac{1}{4}} \div g\,\frac{\upsilon i}{2};$$
where $\varepsilon^{\beta\upsilon^2} = \varepsilon^{\frac{\pi\upsilon}{\omega}} = q^{-1}$. It is obvious that the relation between q and q_1 is $lq\,.\,lq_1 = -\,\pi^2$.

We obtain from the above
$$g\,\frac{\upsilon i}{2}\,\mathfrak{G}\,\frac{\upsilon i}{2} = q^{-\frac{1}{4}} \dots\dots\dots\dots (26).$$

Also, by making $x = \frac{\upsilon i}{2}$ in the expression for $\gamma\left(x + \frac{\omega}{2}\right)$ and $x = \frac{\omega}{2}$ in that for $\gamma\left(x + \frac{\upsilon i}{2}\right)$, we have
$$\gamma\left(\frac{\omega}{2}\right) g\left(\frac{\upsilon i}{2}\right) = -\,i\,\gamma\,\frac{\upsilon i}{2}\,G\,\frac{\omega}{2} \dots\dots (27),$$
and the same or an equivalent one would have been obtained from the functions g, G, $\mathfrak{G}$.

By combining the above systems, we deduce one of the form
$$\gamma\left(x + \frac{\omega}{2} + \frac{\upsilon i}{2}\right) = \varepsilon^{\frac{1}{2}\beta x(\omega - \upsilon i)}\,A''\,\mathfrak{G}x \dots\dots (K),$$
$$g\left(x + \frac{\omega}{2} + \frac{\upsilon i}{2}\right) = \varepsilon^{\frac{1}{2}\beta x(\omega - \upsilon i)}\,B''\,Gx,$$
$$G\left(x + \frac{\omega}{2} + \frac{\upsilon i}{2}\right) = \varepsilon^{\frac{1}{2}\beta x(\omega - \upsilon i)}\,C''\,gx,$$
$$\mathfrak{G}\left(x + \frac{\omega}{2} + \frac{\upsilon i}{2}\right) = \varepsilon^{\frac{1}{2}\beta x(\omega - \upsilon i)}\,D''\,\gamma x.$$
And observing the equation $\varepsilon^{\beta\omega\upsilon i} = \varepsilon^{\pi i} = (-\,1)$, with the following values for the coefficients,
$$A'' = (-\,1)^{\frac{1}{4}}\,.\,\gamma\,\frac{\omega}{2}\,g\,\frac{\upsilon i}{2} \dots\dots\dots\dots (L),$$
$$B'' = -\,(-\,1)^{\frac{1}{4}}.\,q_1^{-\frac{1}{4}}\,.\,\gamma\,\frac{\upsilon i}{2} \div \gamma\,\frac{\omega}{2},$$

$$C'' = (-1)^{\frac{1}{4}} . \mathfrak{G}\frac{\omega}{2}\,\mathfrak{G}\frac{\upsilon i}{2},$$

$$D'' = -(-1)^{\frac{1}{4}} . q^{-\frac{1}{4}} . \mathfrak{G}\frac{\omega}{2} \div \gamma\frac{\upsilon i}{2}.$$

Collecting the formulæ with connect $\gamma\left(\frac{\omega}{2}\right).\gamma\left(\frac{\upsilon i}{2}\right)....$ these are

$$g\frac{\omega}{2} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (M).$$

$$G\frac{\upsilon i}{2} = 0,$$

$$G\frac{\omega}{2}\,\mathfrak{G}\frac{\omega}{2} = q_1^{-\frac{1}{4}},$$

$$g\frac{\upsilon i}{2}\,\mathfrak{G}\frac{\upsilon i}{2} = q^{-\frac{1}{4}},$$

$$\gamma\frac{\omega}{2} . g\frac{\upsilon i}{2} = -i\gamma\left(\frac{\upsilon i}{2}\right) . G\left(\frac{\omega}{2}\right).$$

And by the assistance of these

$$B''C'' \div A''D'' = B'D' \div A'C' = CD \div AB = -1 \;\ldots\ldots\ldots (28),$$

$$A'B' \div C'D' = -A''B'' \div C''D'' = -\gamma^2\left(\frac{\upsilon i}{2}\right) \div \mathfrak{G}^2\left(\frac{\upsilon i}{2}\right),$$

$$A''C'' \div B''D'' = -A'C' \div B'D' = \gamma^2\left(\frac{\omega}{2}\right) \div \mathfrak{G}^2\left(\frac{\omega}{2}\right),$$

$$AD \div BC = -A'D' \div B'C' = \gamma^2\left(\frac{\omega}{2}\right) \div G^2\left(\frac{\omega}{2}\right) = -\gamma^2\left(\frac{\upsilon i}{2}\right) \div g^2\left(\frac{\upsilon i}{2}\right),$$

which will be required presently.

It is now easy to proceed to the general systems of formulæ,

$$\Theta = (-1)^{mn} . \varepsilon^{\beta x . (m\omega - n\upsilon i)} . q_1^{-\frac{1}{2}m^2} . q^{-\frac{1}{2}n^2} \ldots (M).$$

$$\gamma\{x + (m, n)\} = (-1)^{m+n} . \Theta\gamma x,$$

$$g\{x + (m, n)\} = (-1)^{m} . \Theta g x,$$

$$G\{x + (m, n)\} = (-1)^{n} . \Theta G x,$$

$$\mathfrak{G}\{x + (m, n)\} = \Theta\mathfrak{G} x.$$

$$\Phi = (-1)^{n(m+\frac{1}{2})} . \varepsilon^{\beta x[(m+\frac{1}{2})\omega - n\upsilon i]}\, q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} . q^{-\frac{1}{2}n^2}.$$

$$\gamma\{x + (\overline{m}, n)\} = (-1)^{m+n} . \Phi A g x,$$

$$g\{x + (\overline{m}, n)\} = (-1)^{m} . \Phi B\gamma x,$$

$$G\{x + (\overline{m}, n)\} = (-1)^{n} . \Phi C\mathfrak{G} x,$$

$$\mathfrak{G}\{x + (\overline{m}, n)\} = \Phi D G x.$$

$$\Psi = (-1)^{m(n+\frac{1}{2})} \cdot \epsilon^{\beta x \cdot [m\omega - (n+\frac{1}{2})vi]}\, q_1^{-\frac{1}{2}m^2} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$$\begin{aligned}
\gamma\{x + (m, \bar{n})\} &= (-1)^{m+n} \cdot \Psi A' G x, \\
g\{x + (m, \bar{n})\} &= (-1)^{m} \cdot \Psi B' \mathfrak{G} x, \\
G\{x + (m, \bar{n})\} &= (-1)^{n} \cdot \Psi C' \gamma x, \\
\mathfrak{G}\{x + (m, \bar{n})\} &= \Psi D' g x.
\end{aligned}$$

$$\Omega = (-1)^{mn + \frac{m}{2} + \frac{n}{2}} \cdot \epsilon^{\beta x [(m+\frac{1}{2})\omega - (n+\frac{1}{2})vi]} \cdot q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$$\begin{aligned}
\gamma\{x + (\bar{m}, \bar{n})\} &= (-1)^{m+n} \cdot \Omega A'' \mathfrak{G} x, \\
g\{x + (\bar{m}, \bar{n})\} &= (-1)^{m} \cdot \Omega B'' G x, \\
G\{x + (\bar{m}, \bar{n})\} &= (-1)^{n} \cdot \Omega C'' g x, \\
\mathfrak{G}\{x + (\bar{m}, \bar{n})\} &= \cdot \Omega D'' \gamma x.
\end{aligned}$$

Suppose $x = 0$, we have the new systems,

$$\Theta_0 = (-1)^{mn} \cdot q_1^{-\frac{1}{2}m^2} q^{-\frac{1}{2}n^2} \ldots\ldots (M\, bis).$$

$$\begin{aligned}
\gamma(m, n) &= 0, & \gamma'(m, n) &= (-1)^{m+n}\, \Theta_0, \\
g(m, n) &= (-1)^m \cdot \Theta_0, \\
G(m, n) &= (-1)^n \cdot \Theta_0, \\
\mathfrak{G}(m, n) &= \Theta_0.
\end{aligned}$$

$$\Phi_0 = (-1)^{n \cdot (m+\frac{1}{2})} q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} q^{-\frac{1}{2}n^2}.$$

$$\begin{aligned}
\gamma(\bar{m}, n) &= (-1)^{m+n} \cdot \Phi_0 \cdot A, \\
g(\bar{m}, n) &= 0, & g'(\bar{m}, n) &= (-1)^m \cdot \Phi_0 B, \\
G(\bar{m}, n) &= (-1)^n \cdot \Phi_0 C, \\
\mathfrak{G}(\bar{m}, n) &= \Phi_0 D.
\end{aligned}$$

$$\Psi_0 = (-1)^{m \cdot (n+\frac{1}{2})} q_1^{-\frac{1}{2}m^2} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$$\begin{aligned}
\gamma(m, \bar{n}) &= (-1)^{m+n} \cdot \Psi_0 A', \\
g(m, \bar{n}) &= (-1)^m \cdot \Psi_0 B', \\
G(m, \bar{n}) &= 0, & G'(m, \bar{n}) &= (-1)^n \cdot \Psi_0 C', \\
\mathfrak{G}(m, \bar{n}) &= \Psi_0 D'.
\end{aligned}$$

$$\Omega_0 = (-1)^{mn + \frac{1}{2}m + \frac{1}{2}n} q_1^{-\frac{1}{2}m^2 - \frac{1}{2}m} \cdot q^{-\frac{1}{2}n^2 - \frac{1}{2}n}.$$

$$\begin{aligned}
\gamma(\bar{m}, \bar{n}) &= (-1)^{m+n} \cdot \Omega_0 A'', \\
g(\bar{m}, \bar{n}) &= (-1)^m \cdot \Omega_0 B'', \\
G(\bar{m}, \bar{n}) &= (-1)^n \cdot \Omega_0 C'', \\
\mathfrak{G}(\bar{m}, \bar{n}) &= 0, & \mathfrak{G}'(\bar{m}, \bar{n}) &= \Omega_0 D''.
\end{aligned}$$

We obtain immediately, by taking the logarithmic differentials of the functions γx, gx, Gx, $\mathfrak{G}x$, the equations

$$\begin{aligned}
\gamma' x \div \gamma x &= \Sigma\Sigma\,\{x - (m,\, n)\}^{-1}, \quad m = 0,\ n = 0 \text{ admissible},\\
g'x \div gx &= \Sigma\Sigma\,\{x - (\bar{m},\, n)\}^{-1}, \ldots\ldots\ldots\ldots (N),\\
G'x \div Gx &= \Sigma\Sigma\,\{x - (m,\, \bar{n})\}^{-1},\\
\mathfrak{G}'x \div \mathfrak{G}x &= \Sigma\Sigma\,\{x - (\bar{m},\, \bar{n})\}^{-1},
\end{aligned}$$

the limits being the same as in the case of the factorial expressions.

Consider an equation

$$gx\,.\,Gx \div \gamma x\,.\,\mathfrak{G}x = \Sigma\Sigma\,[\mathfrak{A}\,\{x - (m,\, n)\}^{-1} + \mathfrak{B}\{x - (\bar{m},\, \bar{n})\}^{-1}] \ldots (29),$$

we have

$$\mathfrak{A} = g\,(m, n) G(m, n) \div \gamma'\,(m, n)\,\mathfrak{G}(m, n) = 1 \ldots\ldots\ldots\ldots\ldots (30),$$

$$\mathfrak{B} = g\,(\bar{m}, \bar{n}) G(\bar{m}, \bar{n}) \div \gamma(\bar{m}, \bar{n})\,\mathfrak{G}'(\bar{m}, \bar{n}) = B'' C'' \div A'' D'' = -1 \ldots (31).$$

(The application of the ordinary method of decomposition into partial fractions, which is in general exceedingly precarious when applied to transcendental functions, is justified here by a theorem of Cauchy's, which will presently be quoted.) We have thus

$$gx\, Gx \div \gamma x\, \mathfrak{G}x = (\gamma' x \div \gamma x) - (\mathfrak{G}'x \div \mathfrak{G}x),$$

and similarly

$$\begin{aligned}
gx\, \mathfrak{G}x \div \gamma x\, Gx &= (\gamma' x \div \gamma x) - (G'x \div Gx), \ldots . (O),\\
Gx\, \mathfrak{G}x \div \gamma x\, gx &= (\gamma' x \div \gamma x) - (g'x \div gx),\\
-\, b^2\, \gamma x\, \mathfrak{G}x \div gx\, Gx &= (g'x \div gx) - (G'x \div Gx),\\
e^2\, \gamma x\, gx \div Gx\, \mathfrak{G}x &= (G'x \div Gx) - (\mathfrak{G}'x \div \mathfrak{G}x),\\
c^2\, \gamma x\, Gx \div \mathfrak{G}x\, gx &= (\mathfrak{G}'x \div \mathfrak{G}x) - (g'x \div gx);
\end{aligned}$$

in which we have written

$$\gamma\, \frac{\upsilon i}{2} \div \mathfrak{G}\left(\frac{\upsilon i}{2}\right) = \frac{i}{e} \ldots\ldots\ldots\ldots (32),$$

$$\gamma\left(\frac{\omega}{2}\right) \div \mathfrak{G}\left(\frac{\omega}{2}\right) = \frac{1}{c},$$

$$\gamma\left(\frac{\omega}{2}\right) \div G\left(\frac{\omega}{2}\right) = -\, i\left(\gamma\, \frac{\upsilon i}{2} \div g\, \frac{\upsilon i}{2}\right) = \frac{1}{b}\,.$$

Eliminating the derived coefficients,

$$\begin{aligned}
G^2 x - \mathfrak{G}^2 x &= \quad e^2 \gamma^2 x, \ldots\ldots\ldots (33),\\
g^2 x - G^2 x &= -\, b^2 \gamma^2 x,\\
\mathfrak{G}^2 x - g^2 x &= \quad c^2 \gamma^2 x.
\end{aligned}$$

Adding $b^2 = e^2 + c^2$ or $b = \sqrt{(e^2 + c^2)}$, in which sense it will continue to be used.

Also, $$g^2x = \mathfrak{G}^2x - c^2\gamma^2x, \ldots\ldots\ldots\ldots (P),$$
$$G^2x = \mathfrak{G}^2x + e^2\gamma^2x.$$

Suppose $$\phi x = \gamma x \div \mathfrak{G}x, \ldots\ldots\ldots\ldots (Q).$$
$$fx = gx \div \mathfrak{G}x,$$
$$Fx = Gx \div \mathfrak{G}x.$$

Then $$f^2x = 1 - c^2\phi^2x, \ldots\ldots\ldots\ldots (R),$$
$$F^2x = 1 + e^2\phi^2x,$$

and also $$\phi'x = \quad fx\, Fx, \ldots\ldots\ldots\ldots (S),$$
$$f'x = -\, c^2\phi x\, Fx,$$
$$F'x = \quad e^2\phi x\, fx.$$

Whence, putting for fx, Fx, their values

$$1 = \frac{\phi' x}{\sqrt{(1 - c^2\phi^2x)(1 + e^2\phi^2x)}} \ldots . (T);$$

or writing $\phi x = y$, and integrating,

$$x = \int_0^{\phi x} \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}} \ldots . (U),$$

or $$\phi^{-1}y = \int_0^y \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}},$$

which shows that ϕ is an inverse elliptic function.

The equations which are the foundation of the theory of the functions ϕ, f, F, are deduced immediately from the equations (S). (*Abel Œuvres*, tom. I. p. 143.) These are

$$\phi\,(x + y) = \frac{\phi x\, fy\, Fy + \phi y\, fx\, Fx}{1 + e^2c^2\,\phi^2x\,\phi^2y} \ldots . (V),$$

$$f(x + y) = \frac{fx\, fy - c^2\,\phi x\,\phi y\, Fx\, Fy}{1 + e^2c^2\,\phi^2x\,\phi^2y},$$

$$F(x + y) = \frac{Fx\, Fy + e^2\,\phi x\,\phi y\, fx\, fy}{1 + e^2c^2\,\phi^2x\,\phi^2y};$$

so that from this point we may take for granted any properties of these functions. We see, for instance, immediately, $\phi\,\frac{vi}{2} = \frac{i}{e}$, $\phi\left(\frac{\omega}{2}\right) = \frac{1}{c}$; whence

$$\frac{\omega}{2} = \int_0^{\frac{1}{c}} \frac{dy}{\sqrt{(1 - c^2y^2)(1 + e^2y^2)}}, \ldots . (W),$$

$$\frac{vi}{2} = \int_0^{\frac{i}{e}} \frac{dy}{\sqrt{(1-c^2y^2)(1+e^2y^2)}}, \quad \text{or} \quad \frac{v}{2} = \int_0^{\frac{1}{e}} \frac{dy}{\sqrt{(1+c^2y^2)(1-e^2y^2)}} \quad \ldots\ldots (X),$$

which give the values of ω, v in terms of c, e; values which may be developed in a variety of ways, in infinite series. We may also express $\gamma \frac{\omega}{2}$, &c., and consequently $A, B\ldots$&c., by means of the quantities c, e. We have only to combine the equations

$$\gamma\left(\frac{\omega}{2}\right) \div \mathfrak{G}\left(\frac{\omega}{2}\right) = \frac{1}{c}, \quad \gamma\left(\frac{vi}{2}\right) \div \mathfrak{G}\frac{vi}{2} = \frac{i}{e},$$

$$G\frac{\omega}{2} \div \mathfrak{G}\frac{\omega}{2} = \frac{b}{c}, \quad g\frac{vi}{2} \div \mathfrak{G}\frac{vi}{2} = \frac{b}{e} \ldots\ldots (34),$$

with the former relations between these quantities, and we have

$$\gamma\frac{\omega}{2} = b^{-\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \qquad \gamma\frac{vi}{2} = ib^{-\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}, \ldots\ldots (Y),$$

$$g\frac{\omega}{2} = 0, \qquad g\frac{vi}{2} = b^{\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}.$$

$$G\frac{\omega}{2} = b^{\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \qquad \mathfrak{G}\frac{vi}{2} = 0,$$

$$\mathfrak{G}\frac{\omega}{2} = b^{-\frac{1}{2}} c^{\frac{1}{2}} q_1^{-\frac{1}{8}}, \qquad \mathfrak{G}\frac{vi}{2} = b^{-\frac{1}{2}} e^{\frac{1}{2}} q^{-\frac{1}{8}}.$$

$$A = b^{-\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad A' = ib^{-\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}, \quad A'' = (-1)^{\frac{1}{4}} c^{-\frac{1}{2}} e^{-\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}}, \ldots\ldots (Z).$$

$$B = -b^{\frac{1}{2}} c^{\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad B' = b^{\frac{1}{2}} e^{-\frac{1}{2}} q^{-\frac{1}{8}}, \quad B'' = -(-1)^{\frac{1}{4}} ic^{\frac{1}{2}} e^{-\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}},$$

$$C = b^{\frac{1}{2}} c^{-\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad C' = -ib^{\frac{1}{2}} e^{\frac{1}{2}} q^{-\frac{1}{8}}, \quad C'' = (-1)^{\frac{1}{4}} c^{-\frac{1}{2}} e^{\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}},$$

$$D = b^{-\frac{1}{2}} c^{\frac{1}{2}} q_1^{-\frac{1}{8}}, \quad D' = b^{-\frac{1}{2}} e^{\frac{1}{2}} q^{-\frac{1}{8}}, \quad D'' = -(-1)^{\frac{1}{4}} ic^{\frac{1}{2}} e^{\frac{1}{2}} q_1^{-\frac{1}{8}} q^{-\frac{1}{8}},$$

which are to be substituted in any formulæ into which these quantities enter.

The following is Cauchy's Theorem, (*Exercises de Math.* tom. II. p. 289).

"If in attributing to the modulus r of the variable

$$z = r\{\cos p + \sqrt{(-1)} \sin p\} \ldots\ldots (35),$$

infinitely great values, these can be chosen so that the two functions

$$\frac{fz + f(-z)}{2}, \quad \frac{fz - f(-z)}{2z}, \ldots. (36),$$

sensibly vanish, whatever be the value of p, or vanish in general, though ceasing to do so, and obtaining *finite* values for certain particular values of p; then

$$fx = \mathcal{E} \frac{\{(fz)\}}{x-z} \ldots\ldots\ldots\ldots (37),$$

the integral residue being reduced to its principal value."

To understand this, it is only necessary to remark that the integral residue in question is the series of fractions that would be obtained by the ordinary process of decomposition; and by the principal value is meant, that *all* those roots are to be taken, the modulus of which is not greater than a certain limit, this limit being afterwards made infinite.

Suppose now fx is a fraction, the numerator and denominator of which are monomials of the form $(\gamma x)^l (gx)^m \ldots$, $l, m \ldots$ being positive integers, and of course no common factor being left in the numerator and denominator.

Let λ be the excess of the degree of the denominator over that of the numerator. Suppose the modulus r of (z) has any value not the same with any of the moduli of

$$(m, n), \ (\bar{m}, n), \ (m, \bar{n}), \ (\bar{m}, \bar{n}) \ldots. (38).$$

Then we have

$$r(\cos p + i \sin p) = m\omega + nvi + \theta \ldots (39),$$

θ being a finite quantity, such that none of the functions $J\theta$ vanish. m and n are the greatest integer values which allow the possible part of θ and the coefficient of its impossible part to remain positive. We have therefore

$$m^2\omega^2 + n^2v^2 = r^2 - M \ldots\ldots\ldots (40),$$

M being finite; or when r is infinite, at least one of the values m, n is infinite. The function fz reduces itself to the form

$$q_1^{\overline{\frac{\lambda m^2}{2}}} q^{\overline{\frac{\lambda n^2}{2}}} \varepsilon^{mA + nB} F \ldots\ldots\ldots (41),$$

where F is finite. Hence q_1 and q being always less than unity, fz, and consequently both $\frac{1}{2}\{fz + f(-z)\}$ and $\frac{1}{2z}\{fz - f(-z)\}$ vanish for $r = \infty$, as long as λ is positive.

In the case of $\lambda = 0$, the conditions are still satisfied, if we suppose fx to denote an uneven function of x: for when $\lambda=0$, the index of exponential in the above expression vanishes,

or fz is constantly finite. But fz being an odd function of z, $fz + f(-z) = 0$. And $\frac{1}{2z}\{fz - f(-z)\}$ vanishes for z infinite, on account of the z in the denominator: hence the expansion is admissible in this case. But it is certainly so also, in a great many cases at least, where fz is an even function of z; for these may be deduced from the others by a simple change in the value of the variable. For instance, from the expansion of $\gamma x \div gx$, which is an odd function, by writing $x + \frac{\upsilon i}{2}$ for x, we obtain that of $Gx \div \mathfrak{G}x$, which is even.

A case of some importance is when the function is of the above form, multiplied by an exponential $\varepsilon^{\frac{1}{2}ax^2+bx}$. Here writing $z = m\omega + n\upsilon i + \theta$, the admissibility of the formula depends on the evanescence of

$$\varepsilon^{\frac{1}{2}a(m\omega+n\upsilon i)^2}\, q_1^{\frac{1}{2}\lambda m^2}\, q^{\frac{1}{2}\lambda n^2} \;\ldots\ldots (42);$$

or, if $a = h + ki$, this becomes, omitting a finite factor,

$$\varepsilon^{-\frac{1}{2}m^2(\lambda\beta - h)\omega^2 - \frac{1}{2}n^2(\lambda\beta + h)\upsilon^2 - kmn\,\omega\upsilon} \;\ldots (43),$$

which vanishes if $h^2 + k^2 < \lambda^2\beta^2$, i.e. the modulus of (a) is less than $\lambda\beta$. The limiting case is admissible when the series is convergent.

We obtain in this way a very great variety of formulæ. For instance,

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div \gamma x$$

$$= \Sigma\Sigma\left[(-1)^{-mn-m-n}\,\varepsilon^{\frac{1}{2}a(m,n)^2+b(m,n)}\, q_1^{\frac{1}{2}m^2}\, q^{\frac{1}{2}n^2}\{x-(m,n)\}^{-1}\right] \cdots (A'),$$

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div gx$$

$$= -b^{\frac{1}{2}}c^{-\frac{1}{2}}\,\Sigma\Sigma\left[(-1)^{-mn-m-\frac{1}{2}n}\,\varepsilon^{\frac{1}{2}a(\overline{m},n)^2+b(\overline{m},n)}\, q_1^{\frac{1}{2}(m+\frac{1}{2})^2}\, q^{\frac{1}{2}n^2}\{x-(\overline{m},n)\}^{-1}\right],$$

$$\varepsilon^{\frac{1}{2}ax^2+bx} \div Gx$$

$$= ib^{-\frac{1}{2}}e^{-\frac{1}{2}}\,\Sigma\Sigma\left[(-1)^{-mn-\frac{1}{2}m-n}\,\varepsilon^{\frac{1}{2}a(m,\overline{n})^2+b(m,\overline{n})}\, q_1^{\frac{1}{2}m^2}\, q^{\frac{1}{2}(n+\frac{1}{2})^2}\{x-(m,\overline{n})\}^{-1}\right],$$

$$\varepsilon^{\frac{1}{2}ax2+bx} \div \mathfrak{G}x$$

$$= ic^{-\frac{1}{2}}e^{-\frac{1}{2}}\,\Sigma\Sigma\left[(-1)^{-(m+\frac{1}{2})(n+\frac{1}{2})}\varepsilon^{\frac{1}{2}a(\overline{m},\overline{n})^2+b(\overline{m},\overline{n})}\, q_1^{\frac{1}{2}(m+\frac{1}{2})^2}\, q^{\frac{1}{2}(n+\frac{1}{2})^2}\{x-(\overline{m},\overline{n})\}^{-1}\right],$$

in which the modulus of a must not exceed β: in the limiting cases, for $a = \beta$, b must be entirely impossible, and for $a = -\beta$, b must be entirely real. The formulæ for γx are

$$\varepsilon^{\frac{1}{2}\beta x^2+bx} \div \gamma x = \Sigma\Sigma(-1)^{-m-n}\, q^{n^2}\,\varepsilon^{b(m,n)}\{x-(m,n)\}^{-1} \ldots (44),$$

$$\varepsilon^{-\frac{1}{2}\beta x^2+bx} \div \gamma x = \Sigma\Sigma(-1)^{m-n}\, q_1^{m^2}\,\varepsilon^{b(m,n)}\{x-(m,n)\}^{-1};$$

and for $b = 0$,

$$\varepsilon^{\frac{1}{2}\beta x^2} \div \gamma x = \Sigma\Sigma(-1)^{-m-n}\, q^{n^2}\{x-(m,n)\}^{-1} \;\ldots\ldots (45),$$

$$\varepsilon^{-\frac{1}{2}\beta x^2} \div \gamma x = \Sigma\Sigma(-1)^{-m-n}\, q_1^{m^2}\{x-(m,n)\}^{-1}.$$

Next the system,

$$\begin{aligned}
\mathfrak{G}x \div \gamma x &= \Sigma\Sigma\,(-1)^{m+n}\,\{x-(m,n)\}^{-1} \;\ldots\ldots\; (B'). \\
gx \div \gamma x &= \Sigma\Sigma\,(-1)^{m}\,\{x-(m,n)^{-1}, \\
\mathfrak{G}x \div \gamma x &= \Sigma\Sigma\,(-1)^{n}\,\{x-(m,n)\}^{-1};
\end{aligned}$$

$$\begin{aligned}
\gamma x \div gx &= -\,b^{-1}\,c^{-1}\;\Sigma\Sigma\,(-1)^{n}\,\{x-(\bar{m},n)\}^{-1}, \\
Gx \div gx &= -\,c^{-1}\;\Sigma\Sigma\,(-1)^{m+n}\,\{x-(\bar{m},n)\}^{-1}, \\
\mathfrak{G}x \div gx &= -\,b^{-1}\;\Sigma\Sigma\,(-1)^{m}\,\{x-(\bar{m},n)\}^{-1};
\end{aligned}$$

$$\begin{aligned}
\gamma x \div Gx &= -\,b^{-1}\,e^{-1}\;\Sigma\Sigma\,(-1)^{m}\,\{x-(m,\bar{n})\}^{-1}, \\
gx \div Gx &= \quad i\,e^{-1}\;\;\Sigma\Sigma\,(-1)^{m+n}\,\{x-(m,\bar{n})\}^{-1}, \\
\mathfrak{G}x \div Gx &= \quad i\,b^{-1}\;\;\Sigma\Sigma\,(-1)^{n}\,\{x-(m,\bar{n})\}^{-1};
\end{aligned}$$

$$\begin{aligned}
\gamma x \div \mathfrak{G}x &= \quad i\,c^{-1}\,e^{-1}\;\Sigma\Sigma\,(-1)^{m+n}\,\{x-(\bar{m},\bar{n})\}^{-1}, \\
gx \div \mathfrak{G}x &= \qquad e^{-1}\;\;\Sigma\Sigma\,(-1)^{m}\,\{x-(\bar{m},\bar{n})\}^{-1}, \\
Gx \div \mathfrak{G}x &= \quad i\,c^{-1}\;\;\Sigma\Sigma\,(-1)^{n}\,\{x-(\bar{m},\bar{n})\}^{-1}.
\end{aligned}$$

which is partially given by Abel.

We may obtain, in like manner, expressions for the functions

$$\frac{1}{\gamma x\, gx},\ \frac{1}{\gamma x\, Gx},\ \ldots \text{ (six terms of this form)} \;\ldots\; (C'),$$

$$\frac{Gx}{\gamma x\, gx},\ \ldots \qquad \text{(twelve)} \qquad \ldots\; (D'),$$

$$\frac{\gamma x\, gx}{\mathfrak{G}x\, Gx},\ \ldots \qquad \text{(six)} \qquad \ldots\; (E'),$$

$$\frac{1}{\gamma x\, gx\, Gx},\ \ldots \qquad \text{(four)} \qquad \ldots\; (F'),$$

$$\frac{\mathfrak{G}x}{\gamma x\, gx\, Gx},\ \ldots \qquad \text{(four)} \qquad \ldots\; (G'),$$

$$\frac{1}{\gamma x\, gx\, Gx\, \mathfrak{G}x},\ \ldots \qquad \text{(one)} \qquad \ldots\; (H');$$

each of them, except (E'), (the system for which, admitting no exponential, has already been given,) multiplied by an exponential $\varepsilon^{\frac{1}{2}ax^2+bx}$, the limits of (a) being $\pm 2\beta$, $\pm\beta$, $-$, $\pm 3\beta$, $\pm 2\beta$, $\pm 4\beta$. For the limiting values, b must be entirely impossible for the superior limit, and entirely possible for the inferior one.

Thus the last case is

$$\frac{1}{\gamma x \, gx \, Gx \, \mathfrak{G}x} \, \varepsilon^{\frac{1}{2}ax^2+bx} \, \dots\dots\dots\dots (H'),$$

$$= \Sigma\Sigma \left[\varepsilon^{\frac{1}{2}a\,(m,\,n)^2+b\,(m,\,n)} \, q_1^{\,2m^2} \, q^{2n^2} \left\{x - (m, n)\right\}^{-1}\right]$$

$$- \Sigma\Sigma \left[\varepsilon^{\frac{1}{2}a\,(\bar{m},\,n)^2+b\,(\bar{m},\,n)} \, q_1^{\,2(m+\frac{1}{2})^2} \, q^{2n^2} \left\{x - (\bar{m}, n)\right\}^{-1}\right]$$

$$+ \Sigma\Sigma \left[\varepsilon^{\frac{1}{2}a\,(m,\,\bar{n})^2+b\,(m,\,\bar{n})} \, q_1^{\,2m^2} \, q^{2(n+\frac{1}{2})^2} \left\{x - (m, \bar{n})\right\}^{-1}\right]$$

$$- \Sigma\Sigma \left[\varepsilon^{\frac{1}{2}a\,(\bar{m},\,\bar{n})^2+b\,(\bar{m},\,\bar{n})} \, q_1^{\,2(m+\frac{1}{2})^2} \, q^{2(n+\frac{1}{2})^2} \left\{x - (\bar{m}, \bar{n})\right\}^{-1}\right];$$

in particular

$$\frac{1}{\gamma x \, gx \, Gx \, \mathfrak{G}x} \, \varepsilon^{2\beta x^2} = \Sigma\Sigma \, q_1^{\,4m^2} \left\{x - (m, n)\right\}^{-1} \dots . (46),$$

$$- \Sigma\Sigma \, q_1^{\,(2m+1)^2} \left\{x - (\bar{m}, n)\right\}^{-1}$$

$$+ \Sigma\Sigma \, q_1^{\,4m^2} \left\{x - (\bar{m}, n)\right\}^{-1}$$

$$+ \Sigma\Sigma \, q_1^{\,(2m+1)^2} \left\{x - (\bar{m}, \bar{n})\right\}^{-1},$$

or the analogous formula obtained by changing β, q_1, m into $-\beta$, q, n.

The function $\phi^2 x$, which is even, and for which $\lambda = 0$, cannot be expanded entirely in a series of partial fractions; but $(x - a)^{-1} \phi^2 x$ may be so expanded. Multiply by $(x - a)$, the second side has for its general term

$$(x - a)(Mx + N)\left\{x - (\bar{m}, \bar{n})\right)^{-2},$$

equivalent to

$$K' + (M'x + N')\left\{x - (\bar{m}, \bar{n})\right\}^{-2}.$$

Summing all the K''s, we have an equation of the form

$$\phi^2 x = A + \Sigma\Sigma \left[L \left\{x - (\bar{m}, \bar{n})\right\}^{-2} + M \left\{x - (\bar{m}, \bar{n})\right\}^{-1}\right] \dots (47).$$

To determine the coefficients as simply as possible, change x into $x + \frac{1}{2}\omega + \frac{1}{2}n\omega i$,

$$- e^{-2} c^{-2} \, \overline{\phi x}|^{-2} = A + \Sigma\Sigma \left[L \left\{x - (m, n)\right\}^{-2} + M \left\{x - (m, n)\right\}^{-1}\right]$$

$$\dots , (48),$$

$$L = - e^{-2} c^{-2} \left[\left\{x - (m, n)\right\}^2 \, \overline{\phi x}|^{-2}\right] \dots\dots\dots\dots (49),$$

$$M = - e^{-2} c^{-2} \, d_x \left[\left\{x - (m, n)\right\}^2 \, \overline{\phi x}|^{-2}\right], \quad x = (m, n).$$

Or writing $x + (m, n)$ for x,

$$L = - e^{-2} c^{-2} \left(x^2 \overline{\phi x}|^{-2}\right), \quad (x = 0), \qquad L = e^{-2} c^{-2} \dots (50),$$

$$M = - e^{-2} c^{-2} \, d_x \left(x^2 \overline{\phi x}|^{-2}\right) = 0\,;$$

$$\phi^2 x = \quad A - e^{-2} c^{-2} \, \Sigma\Sigma \left\{x - (\bar{m}, \bar{n})\right\}^{-2} \dots\dots\dots\dots (51).$$

Integrating twice,

$$\int_0 dx \int_0 dx \,.\, \phi^2 x = \tfrac{1}{2} A x^2 + e^{-2} c^{-2}\, \Sigma\Sigma\, l\,\{x - (\overline{m}, \overline{n})\} \ldots\ldots (52),$$

or

$$\mathfrak{G}x = \varepsilon^{-\frac{1}{2} e^2 c^2 A\, x^2 + e^2 c^2 \int_0 dx \int_0 dx \,.\, \phi^2 x} \ldots\ldots\ldots (53),$$

an equation from which it is easy to determine the coefficient A.

Suppose for a moment $\phi_{,} x = \int_0 \phi^2 x \, dx$, $\phi_{,,} x = \int_0 \phi_{,} x \, dx$; then, since $\phi^2 (x + \omega) - \phi^2 x = 0$,

$$\phi_{,} (x + \omega) = \phi x = \phi_{,} \omega, \quad \phi_{,,} (x + \omega) - \phi_{,,} x = \phi_{,,} \omega + x \phi_{,} \omega.$$

But similarly $\phi^2 x - \phi^2 (\omega - x) = 0$; whence

$$\phi_{,} x + \phi_{,} (\omega - x) = \phi_{,} \omega, \quad \phi_{,,} x - \phi_{,,} (\omega - x) + \phi_{,,} \omega = x \phi_{,} \omega;$$

whence, writing $x = \dfrac{\omega}{2}$,

$$\phi_{,} \omega = 2\phi_{,} \frac{\omega}{2}, \quad \phi_{,,} \omega = \omega . \phi_{,} \frac{\omega}{2}, \text{ or } \phi_{,,} (x + \omega) - \phi_{,,} x = \phi_{,} \left(\frac{\omega}{2}\right) (2x + \omega).$$

Hence

$$\mathfrak{G} (x + \omega) = \varepsilon^{-\frac{1}{2} e^2 c^2 \left(A - \frac{\omega}{2} \phi_{,} \frac{\omega}{2}\right) (2\omega x + \omega^2)}\, \mathfrak{G} x \ldots\ldots (54).$$

But

$$\mathfrak{G} (x + \omega) = \varepsilon^{\beta \omega x} q_1^{-\frac{1}{2}}\, \mathfrak{G} x = \varepsilon^{\frac{\beta}{2} (2\omega x + \omega^2)}\, \mathfrak{G} x \ldots\ldots (55);$$

or, comparing these,

$$- e^2 c^2 \left(A - \frac{2}{\omega} \phi_{,} \frac{\omega}{2}\right) = \beta \ldots\ldots\ldots\ldots (56),$$

$$- \tfrac{1}{2} e^2 c^2 A = \tfrac{1}{2} \beta - \frac{e^2 c^2}{\omega} \phi_{,} \frac{\omega}{2} \ldots\ldots\ldots (57),$$

or writing

$$M = \frac{e^2 c^2}{\omega} \int_0^{\frac{\omega}{2}} \phi^2 x \, dx \ldots\ldots\ldots\ldots (58),$$

$$\mathfrak{G} x = \varepsilon^{(\frac{1}{2}\beta - M)\, x^2 + e^2 c^2 \int_0 dx \int_0 dx \,.\, \phi^2 x} \ldots\ldots (I');$$

which is the formulæ corresponding to the one of Jacobi's referred to at the beginning of this paper. Analogous formulæ may be deduced from it by writing $x + \dfrac{\omega}{2}$, or $x + \dfrac{\upsilon i}{2}$, or $x + \dfrac{\omega}{2} + \dfrac{\upsilon i}{2}$, instead of x.

The following formulæ, making the necessary changes of notation, are taken from Jacobi. We have

$$\phi^2 (x + a) - \phi^2 (x - a) = \frac{4 \phi a . f a . F a . \phi x . f x F x}{(1 + e^2 c^2 . \phi^2 a . \phi^2 x)^2} \ldots\ldots (59),$$

whence

$$\int_0 \{\phi^2 (x + a) - \phi^2 . (x - a)\} \, dx = \frac{2 \phi a . f a . F a . \phi^2 x}{1 + e^2 c^2 \phi^2 . a . \phi^2 x} \ldots (60).$$

The first side of which is

$$\int_{-a}.\phi^2.(x+a)\,dx - \int_{a}.\phi^2(x-a)\,dx - 2.\int_0 \phi^2 a.da \ldots (61).$$

Hence, multiplying by e^2c^2, and observing the value of $\mathfrak{G}x$,

$$\frac{\mathfrak{G}'(x+a)}{\mathfrak{G}(x+a)} - \frac{\mathfrak{G}'(x-a)}{\mathfrak{G}(x-a)} - 2\,\frac{\mathfrak{G}'a}{\mathfrak{G}a} = \frac{2e^2c^2.fa.Fa.\phi a.\phi^2 x}{1+e^2c^2\phi^2 a.\phi^2 x} \ldots (62).$$

If in this case we interchange x, a and add,

$$\frac{\mathfrak{G}'x}{\mathfrak{G}x} + \frac{\mathfrak{G}'a}{\mathfrak{G}a} - \frac{\mathfrak{G}'(x+a)}{\mathfrak{G}(x+a)} = e^2c^2.\phi a.\phi x.\phi(a+x) \ldots (63).$$

[By subtracting, we should have obtained an equation only differing from the above in the sign of (a)].

Integrating the last equation but one, with respect to (a),

$$l\mathfrak{G}(x+a) + l\mathfrak{G}(x-a) - 2l\mathfrak{G}x - 2l\mathfrak{G}a = l(1+e^2c^2\phi^2 x\phi^2 a).$$

the integral being taken from $a = 0$. Whence

$$\mathfrak{G}(x+a)\,\mathfrak{G}(x-a) = \mathfrak{G}^2 x\mathfrak{G}^2 a\,(1+e^2c^2\phi^2 x\phi^2 a) \ldots (64).$$

Or $\qquad \mathfrak{G}(x+a)\,\mathfrak{G}(x-a) = \mathfrak{G}^2 x.\mathfrak{G}^2 a + e^2c^2.\gamma^2 x\gamma^2 a,$

whence also

$$\left.\begin{array}{l} \gamma(x+a)\,\gamma(x-a) = \gamma^2 x\mathfrak{G}^2 a - \gamma^2.a\mathfrak{G}^2 x. \\ g(x+a)\,g(x-a) = g^2 x\mathfrak{G}^2 x - c^2.g^2 a.\mathfrak{G}^2 x. \end{array}\right\} (J').$$

$$G(x+a).\,G(x-a) = G^2 x.\mathfrak{G}^2 a + e^2 G^2 a.\mathfrak{G}^2 x.$$

These equations being obtained from the first by the change of x into $x+\frac{\omega}{2}$, $x+\frac{vi}{2}$, $x+\frac{\omega}{2}+\frac{vi}{2}$. They form a most important group of formulæ in the present theory. (By integrating the same formulæ with respect to x, and representing by $\Pi(x, a)$ the integral $\int_0 \frac{-e^2c^2\phi a fa Fa\phi^2 x dx}{1+e^2c^2\phi^2 a\phi^2 x}$, Jacobi obtains

$$\Pi(x, a) = \tfrac{1}{2}l\,\frac{\mathfrak{G}(x-a)}{\mathfrak{G}(x+a)} + x\,\frac{\mathfrak{G}'a}{\mathfrak{G}a}:$$

an equation which conducts him almost immediately to the formulæ for the addition of the argument or of the parameter in the function Π. This, however, is not very closely connected with the present subject. For some formulæ also deduced from (63), by which $\frac{\mathfrak{G}(x-a)\,\mathfrak{G}(y-a)\,\mathfrak{G}(x+y+a)}{\mathfrak{G}(x+a)\,\mathfrak{G}(y+a)\,\mathfrak{G}(x+y-a)}$ is expressed in terms of the function ϕ. S[illegible]bi.

NOTE.—We have

$$\gamma_\beta x = x\Pi\Pi\left(1 + \frac{x}{(m, n)}\right).$$

$$g\beta x = \Pi\Pi\left(1 + \frac{x}{(\overline{m}, n)}\right).$$

the limits of n being $\pm q$, and those of m being $\pm p$, in the first case, and $p, -p-1$, in the second case. Also $\frac{p}{q} = \infty$. We deduce immediately

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) = \left(x + \frac{\omega}{2}\right)\Pi\Pi\left\{1 + \frac{\left(x + \frac{\omega}{2}\right)}{(m, n)}\right\}$$

$$= \Pi\Pi.\left(1 + \frac{x}{(\overline{m}, n)}\right) \div \frac{\omega}{2}\,\Pi\Pi\,\frac{(m, n)}{(\overline{m}, n)};$$

(paying attention to the omission of $(m = 0,\ n = 0)$ in $\gamma^\beta x$, and supposing that this value enters into the numerator of the expression just obtained, but not into its denominator). This is of the form

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) = A\Pi\Pi\left(1 + \frac{x}{(\overline{m}, n)}\right);$$

but the limits are not the same in this product and in $g_\beta x$. In the latter m assumes the value $-p-1$, which it does not in the former. Hence

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) \div g_\beta x = A \div \Pi_n\left(1 + \frac{x}{-(p + \frac{1}{2})\,\omega + nvi}\right).$$

And the above product reduces itself to unity in consequence of all the values assumed by n being indefinitely small compared with the quantity $(p + \frac{1}{2})$; we have therefore

$$\gamma_\beta\left(x + \frac{\omega}{2}\right) = Ag_\beta x \ldots\ldots\ldots\ldots(27),$$

and similar expressions for the remaining functions. To illustrate this further, suppose we had been considering, instead of $\gamma_\beta x$ the function $\gamma_{-\beta} x$, given by the same formulæ, only instead of $\frac{p}{q} = \infty$, $\frac{p}{q} = 0$. We have in this case also

$$\gamma_{-\beta}\left(x + \frac{\omega}{2}\right) \div g_{-\beta} x = A' \div \Pi_n\left(1 + \frac{x}{(-p + \frac{1}{2})\,\omega + nvi}\right).$$

A' different from A on account of the different limits. The divisor of the second side takes the form

$$\{x-(p+\tfrac{1}{2})\omega.\}.\Pi\left(1+\frac{x-(p+\frac{1}{2})\omega}{nvi}\right)\div(-p+\tfrac{1}{2})\omega.\Pi\left(1-\frac{(p+\frac{1}{2})\omega}{nvi}\right),$$

and the extreme values of n being infinite as compared with p. This may be reduced to

$$-\sin\frac{\pi}{vi}\{x-(p+\tfrac{1}{2})\omega\}\div\sin(p+\tfrac{1}{2})\omega.$$

$$=\varepsilon^{\frac{\pi}{v}[(p+\frac{1}{2})\omega-x]}\div\varepsilon^{\frac{\pi}{v}[(p+\frac{1}{2})\omega]}=\varepsilon^{-\frac{\pi x}{v}};$$

neglecting the exponentials whose indices are infinitely great and negative. Observing the value of β this becomes $\varepsilon^{-\omega\beta x}$, and we have

$$\gamma_{-\beta}\left(x+\frac{\omega}{2}\right)=\varepsilon^{\beta\omega x}.A'g_{-\beta}.x:$$

a result of the form of that which would be deduced from the equations $\gamma_{-\beta}x=\varepsilon^{\beta x^2}\gamma_\beta x$, $g_{-\beta}x=\varepsilon^{\beta x^2}g_\beta x$, $\gamma_\beta\left(x+\dfrac{\omega}{2}\right)=Ag\beta x.$ It is scarcely necessary to remark that $\gamma_{-\beta}x$ has the same relations to the change of x into $x+\dfrac{vi}{2}$ as $\gamma_\beta x$ has to that of x into $x+\dfrac{\omega}{2}$.

VI.—ON BRIANCHON'S HEXAGON.

By Percival Frost, M.A., St. John's College.

The following is a proof of Brianchon's property of the hexagon circumscribing a conic section, in which the method of multiplication is used, which I made use of to solve a problem in transversals in No. xxi. p. 113. The property is stated in the last number, and proved for each case separately.

Let the conic sections be referred to two opposite sides of the circumscribing hexagon as axes, and let the equation to the curve be $\quad ax+by-1=2B\sqrt{(xy)}.$

Then, if $\alpha x+\beta y=1$ be the equation to a tangent,

$$ax+by-(\alpha x+\beta y)=2B\sqrt{(xy)}$$

must give equal values to $\sqrt{\dfrac{y}{x}}$;

hence $\qquad (a-\alpha)(b-\beta)=B^2\ \ldots\ldots\ldots\ (1).$

Let AB, CD (see fig. 2) be the two sides taken as axes, AE, EC and BF, FD pairs of contiguous sides.

Let the equations be

$$\left.\begin{array}{ll} ax + \beta' y = 1 & \text{to } AE \\ a'x + \beta y = 1 & \text{" } EC \end{array}\right\} \dots\dots\dots (2),$$

$$\left.\begin{array}{ll} \gamma x + \delta' y = 1 & \text{" } BF \\ \gamma' x + \delta y = 1 & \text{" } FD \end{array}\right\} \dots\dots\dots (3),$$

multiply equations (2) by h and k respectively, and add; therefore at E we have the relation

$$(ha + ka')\,x + (h\beta' + k\beta)\,y = h + k.$$

Similarly at F we have the relation

$$(h'\gamma + k'\gamma')\,x + (h'\delta' + k'\delta)\,y = h' + k' \dots . (4);$$

and if the arbitrary multipliers be so chosen as to make these coincide, either will be the equation to the said line EF. In this case

$$\begin{aligned} ha + ka' &= h'\gamma + k'\gamma', \\ h\beta' + k\beta &= h'\delta' + k'\delta, \\ h + k &= h' + k'; \end{aligned}$$

therefore, eliminating by cross multiplication h and k,

$$\left.\begin{array}{l} 0 = k'\left\{\gamma'(\beta - \beta') + \delta(a - a') + a'\beta' - a\beta\right\} \\ \quad + h'\left\{\gamma(\beta - \beta') + \delta'(a - a') + a'\beta' - a\beta\right\} \end{array}\right\} \dots (5).$$

Now by (1) we have the relations

$$(a - a).(b - \beta') = (a - a').(b - \beta) = (a - \gamma').(b - \delta) = \dots$$

whence $$\frac{\beta - \beta'}{a - a'} = \frac{b - \beta}{a - a} \dots\dots\dots\dots (6),$$

and $$\frac{\delta - \beta'}{\gamma' - a} = \frac{\delta - b}{a - a};$$

therefore adding these equations we obtain

$$\frac{\gamma'(\beta - \beta') + \delta(a - a') + a'\beta' - a\beta}{(a - a')(\gamma' - a)} = \frac{\delta - \beta}{a - a}.$$

Similarly, by the symmetry of (1), changing γ and a and a into δ, β and b respectively, and *vice versâ*,

$$\frac{\delta'(a - a') + \gamma(\beta - \beta') + a'\beta' - a\beta}{(\beta - \beta')(\delta' - \beta)} = \frac{\gamma - a}{b - \beta};$$

therefore we have by (5) and (6)

$$0 = k'(\delta - \beta)(\gamma' - a) + h'(\delta' - \beta)(\gamma - a) \dots . (7).$$

Now $\quad \alpha x + \delta y = 1$ is the equation to AD,

$\gamma x + \beta y = 1$ " " " BC;

therefore, multiplying by m and n, and adding,

$$(m\alpha + n\gamma)\,x + (m\delta + n\beta)\,y = m + n \;\ldots .\; (8)$$

is a relation which holds at G, the intersection.

Assume m and n so as to satisfy the equations

$$m\alpha + n\gamma = h'\gamma + k'\gamma',$$
$$m\delta + n\beta = h'\delta' + k'\delta;$$

therefore $\quad m(\gamma\delta - \alpha\beta) = h'(\gamma\delta' - \beta\gamma) + k'(\gamma\delta - \beta\gamma');$

or $\quad (m - k')(\gamma\delta - \alpha\beta) = h'\gamma(\delta' - \beta) + k'\beta(\alpha - \gamma').$

Similarly $(n - h')(\gamma\delta - \alpha\beta) = h'\alpha(\beta - \delta') + k'\delta(\gamma' - \alpha);$

hence $\{m + n - (h' + k')\}\,(\gamma\delta - \alpha\beta)$

$$= h'(\gamma - \alpha)(\delta' - \beta) + k'(\delta - \beta)(\gamma' - \alpha) = 0, \text{ by } (7);$$

and $\quad m + n = h' + k';$

therefore the relations (8) and (4) coincide, or the point G is in the straight line EF; which proves the proposition.

VII.—ON THE LINES OF CURVATURE OF SURFACES OF THE SECOND ORDER.

By William Thomson, B.A. St. Peter's College.

The method usually followed in works on Geometry of Three Dimensions, in treating the differential equation to the lines of curvature of an ellipsoid, leads to an unsymmetrical integral, involving only two of the co-ordinates, and therefore representing the projection of the lines of curvature on one of the co-ordinate planes. In the following paper, by making use of an equivalent process, but preserving the symmetry with respect to the two variables which are involved, an integral is obtained which enables us from the symmetry to infer the equations to the projections on the other two co-ordinate planes. By combining the three forms of the integral thus obtained, we arrive at the integral given by Mr. Ellis, and at other symmetrical formulæ.

Let the equation to the surface be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The differential equation to the lines of curvature is consequently

$$\frac{(b^2 - c^2)\,x}{dx} + \frac{(c^2 - a^2)\,y}{dy} + \frac{(a^2 - b^2)\,z}{dz} = 0.$$

Let $\frac{x^2}{a^2}=u$, $\frac{y^2}{b^2}=v$, $\frac{z^2}{c^2}=w$. The preceding equations become

$$u+v+w=1 \ldots\ldots\ldots\ldots (1),$$

$$\frac{(b^2-c^2)\,u}{du}+\frac{(c^2-a^2)\,v}{dv}+\frac{(a^2-b^2)\,w}{dw}=0.\ldots (2).$$

Eliminating u from the latter equation, by means of the former, we have

$$\frac{b^2-c^2}{du}+v\left(\frac{c^2-a^2}{dv}-\frac{b^2-c^2}{du}\right)+w\left(\frac{a^2-b^2}{dw}-\frac{b^2-c^2}{du}\right)=0,$$

or

$$b^2-c^2=\frac{v\,(-c^2du-c^2dv+a^2du+b^2dv)}{dv}$$
$$-\frac{w\,(-b^2du-b^2dw+a^2du+c^2dw)}{dw}.$$

But $du+dv+dw=0$, by (1); and hence the equation to the lines of curvature may be put under the form

$$\frac{v}{dv}-\frac{w}{dw}=\frac{b^2-c^2}{a^2du+b^2dv+c^2dw}.$$

If from this equation we eliminate du, we obtain an equation of Clairaut's form, of which the integral is found by substituting for $\frac{dv}{dw}$ an arbitrary constant. For the sake of symmetry we shall denote this constant by $\frac{g}{h}$; and we must consequently substitute g and h for dv and dw in the differential equation, and therefore also for du, $-(g+h)$, which we shall denote by f. Thus we have

$$\left.\begin{array}{l}\dfrac{du}{f}=\dfrac{dv}{g}=\dfrac{dw}{h}\\ f+g+h=0,\end{array}\right\}\ldots\ldots\ldots\ldots (3),$$

where

and the complete integral is

$$\frac{v}{g}-\frac{w}{h}=\frac{b^2-c^2}{a^2f+b^2g+c^2h}\ldots\ldots\ldots (4).$$

Also, by the symmetry, we have for the integrals involving the variables wu and uv,

$$\left.\begin{array}{l}\dfrac{w}{h}-\dfrac{u}{f}=\dfrac{c^2-a^2}{a^2f+b^2g+c^2h}\\[2ex] \dfrac{u}{f}-\dfrac{v}{g}=\dfrac{a^2-b^2}{a^2f+b^2g+c^2h}\end{array}\right\}\ldots\ldots\ldots (4),$$

The manner in which the quantities f, g, h have been introduced, shows clearly how they represent only one arbitrary constant. Any one of the equations (4) may be written in such a form as to contain only one arbitrary constant explicitly; and it will be shown below how f, g, h may be expressed symmetrically by two arbitrary constants, one of which is irrelevant, as it enters as a factor in the integral.

From the equations (4), as from the ordinary forms, the properties of the projections of the lines of curvature may be readily deduced. Thus, taking the second, and substituting for w, u, and g their values $\frac{z^2}{c^2}$, $\frac{x^2}{a^2}$, and $-(f+h)$, we have

$$\frac{z^2}{c^2 h} - \frac{x^2}{a^2 f} = \frac{a^2 - c^2}{(b^2 - c^2)h - (a^2 - b^2)f}.$$

Let a^2, b^2, c^2 be positive quantities in descending order of magnitude. Then, unless f and h have opposite signs, this equation cannot be satisfied by any values of z, x which satisfy the inequality

$$\frac{z^2}{c^2} + \frac{x^2}{a^2} < 1;$$

that is, by values which correspond to any point of the ellipsoid. Hence we may write the equation as

$$\frac{z^2}{\gamma^2} + \frac{x^2}{\alpha^2} = 1 \ldots\ldots\ldots\ldots\ldots (5),$$

where
$$\gamma^2 = \frac{c^2(a^2 - c^2)h}{(b^2 - c^2)h - (a^2 - b^2)f},$$

$$\alpha^2 = -\frac{a^2(a^2 - c^2)f}{(b^2 - c^2)h - (a^2 - b^2)f}.$$

Eliminating $f : h$ between these equations, we have

$$\frac{\gamma^2(b^2 - c^2)}{c^2} + \frac{\alpha^2(a^2 - b^2)}{a^2} = a^2 - c^2 \ldots\ldots (6).$$

Hence we conclude that the projections of the lines of curvature on the plane of the greatest and least axes of the ellipsoid, are the ellipses whose semiaxes, γ, α, are connected by the equation (6). Thus the construction for describing them is as follows. Draw an ellipse, concentric with the ellipsoid, in the plane ca, with the lines

$$c\left(\frac{a^2 - c^2}{b^2 - c^2}\right)^{\frac{1}{2}}, \quad a\left(\frac{a^2 - c^2}{a^2 - b^2}\right)^{\frac{1}{2}},$$

as semiaxes. Take any point in this ellipse, draw perpendiculars to the axes, and with the intersections as vertic

describe a concentric ellipse. This will be the projection of a line of curvature. Also, by giving the point assumed in the auxiliary ellipse every possible position in its circumference, we obtain the projections of all the lines of curvature. Similar constructions are applicable to the projections of the lines of curvature on the other two principal planes; and, by taking one or two of the quantities a^2, b^2, c^2 negative, we may extend the rules to hyperboloids of one or of two sheets.

In the case we have taken, of an ellipsoid, and the plane of the greatest and least axes for the plane of projection, each curve intersects the consecutive one, and the locus of these intersections may be found from (5) and (6) by the ordinary process. Thus, by differentiation,

$$\frac{z^2}{\gamma^3}\,d\gamma + \frac{x^2}{\alpha^3}\,d\alpha = 0,$$

$$\frac{a^2-b^2}{a^2}\,\alpha\,d\alpha + \frac{b^2-c^2}{c^2}\,\gamma\,d\gamma = 0.$$

Hence $$\frac{c^2}{b^2-c^2}\,\frac{z^2}{\gamma^4} = \frac{a^2}{a^2-b^2}\,\frac{x^2}{\alpha^4},$$

which gives $$\frac{\gamma^2}{cz\,(b^2-c^2)^{-\frac{1}{2}}} = \frac{\alpha^2}{ax\,(a^2-b^2)^{-\frac{1}{2}}}.$$

By combining this equation, first with (6) and then with (5), we find each member

$$= \frac{a^2-c^2}{\frac{z}{c}\,(b^2-c^2)^{\frac{1}{2}} + \frac{x}{a}\,(a^2-b^2)^{\frac{1}{2}}} = \frac{z}{c}\,(b^2-c^2)^{\frac{1}{2}} + \frac{x}{a}\,(a^2-b^2)^{\frac{1}{2}}.$$

Hence $$\frac{z}{c}\,(b^2-c^2)^{\frac{1}{2}} + \frac{x}{a}\,(a^2-b^2)^{\frac{1}{2}} = (a^2-c^2)^{\frac{1}{2}} \;\ldots\ldots\; (7),$$

is the equation to the required locus, which is therefore a group of four straight lines (on account of the double signs of the radicals) forming a rhombus, of which the diagonals coincide with the axes of c and a.

Thus we see that the projections of the lines of curvature on the plane of the greatest and least axes, are ellipses inscribed in a rhombus, with their axes coincident with those of the ellipsoid. If we consider b^2 as not of intermediate magnitude between a^2 and c^2, the equation (7) represents an imaginary group of straight lines, which shows that the projection of any line of curvature on the plane of the greatest and mean, or of the mean and least axes, does not meet its consecutive.

It may be remarked with respect to equations (4), that any one of them may be deduced from any other, by combining it with the equation $u + v + w = 1$, as is easily verified. Also, by multiplying the first by a^2, the second by b^2, and the third by c^2, and adding, we have

$$\frac{u(b^2 - c^2)}{f} + \frac{v(c^2 - a^2)}{g} + \frac{w(a^2 - b^2)}{h} = 0 \; \ldots\ldots (8),$$

which is the symmetrical integral given by Mr. Ellis, (Vol. II. p. 133). This equation might also have been found directly, when it was proved that

$$\frac{du}{f} = \frac{dv}{g} = \frac{dw}{h};$$

by eliminating by means of these relations, du, dv, dw from (2), the differential equation to the lines of curvature, which is the method followed by Mr. Ellis.

Without losing generality, we may substitute for f, g, h, any expressions in terms of two distinct constants which satisfy the condition $f + g + h = 0$. Thus, if we take k and υ for the constants, we may assume

$$f = ka^2(b^2 - c^2) - k\upsilon(b^2 - c^2),$$

or

$$\left.\begin{aligned} f &= k(b^2 - c^2)(a^2 - \upsilon), \\ g &= k(c^2 - a^2)(b^2 - \upsilon), \\ h &= k(a^2 - b^2)(c^2 - \upsilon), \end{aligned}\right\} \ldots\ldots\ldots\ldots (9).$$

Making these substitutions in (8), we have

$$\left.\begin{aligned} &\frac{u}{a^2 - \upsilon} + \frac{v}{b^2 - \upsilon} + \frac{w}{c^2 - \upsilon} = 0, \\ \text{or} \quad &\frac{x^2}{a^2(a^2 - \upsilon)} + \frac{y^2}{b^2(b^2 - \upsilon)} + \frac{z^2}{c^2(c^2 - \upsilon)} = 0, \end{aligned}\right\} \ldots\ldots (10).$$

Adding this equation, multiplied by υ, to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \; \ldots\ldots\ldots\ldots (a),$$

we have

$$\frac{x^2}{a^2 - \upsilon} + \frac{y^2}{b^2 - \upsilon} + \frac{z^2}{c^2 - \upsilon} = 1 \; \ldots\ldots\ldots (11).$$

This equation shows that the lines of curvature of any surface of the second order with a centre, are its intersections with confocal surfaces of the second order. Since this property is independent of the centre, it follows that it must also be true for the case of a surface without a centre.

If the coordinates x, y, z, of any point in the line of curvature be given, by substituting these values for x, y, z

in (10), which will be a quadratic in v, we may determine two values of this parameter, which, substituted in (11), will give the equations to the hyperboloids of one sheet and of two sheets confocal with the ellipsoid, which cut it in the two lines of curvature passing through the given point x_1, y_1, z_1. In general, equation (11) may be considered as a cubic for determining v, when x, y, z have any given real values whatever, x_1, y_1, z_1. The three roots, which may readily be shown to be real, correspond to the three species of surfaces confocal with the ellipsoid (a, b, c) which intersect in the point x_1, y_1, z_1. In the present case, when this point is on the surface of the ellipsoid (a, b, c), and therefore x_1, y_1, z_1 satisfy the equation (a), one root of the cubic is zero, and the other two are the roots of the quadratic (10), which is the *reduced equation*. Thus, whether we take the form (10) or (11) of the integral, the two arbitrary constants may be determined by the solution of a quadratic equation, from the condition that the curve passes through a given point.

If we wish to determine the direction of the tangent at any point of a line of curvature, we may follow the usual process of differentiation for curves of double curvature.

Thus, taking any two of the equations (4), we find

$$\frac{du}{f} = \frac{dv}{g} = \frac{dw}{h}.$$

But if l, m, n be the direction-cosines of the tangent, we have

$$\frac{l}{\frac{a^2}{x}\,du} = \frac{m}{\frac{b^2}{y}\,dv} = \frac{n}{\frac{c^2}{z}\,dw};$$

and hence

$$\frac{l}{\frac{a^2}{x}f} = \frac{m}{\frac{b^2}{y}g} = \frac{n}{\frac{c^2}{z}h} \quad \ldots\ldots\ldots\ (12).$$

If we substitute for f, g, h their values by (9), these equations become

$$\frac{l}{\frac{a^2}{x}(b^2-c^2)(a^2-v)} = \frac{m}{\frac{b^2}{y}(c^2-a^2)(b^2-v)} = \frac{n}{\frac{c^2}{z}(a^2-b^2)(c^2-v)} \ldots (13).$$

We may also determine l, m, n by the ordinary formulæ for the principal directions of curvature. Thus, if ρ be the radius of curvature of the normal section touching the line of curvature at the point considered, we have

$$l = \frac{\mu x}{a^2 - p\rho}, \quad m = \frac{\mu y}{b^2 - p\rho}, \quad n = \frac{\mu z}{c^2 - p\rho} \ \ldots\ (14);$$

and $p\rho$ is one of the values of Q deduced from the quadratic equation

$$\frac{x^2}{a^2(a^2-Q)}+\frac{y^2}{b^2(b^2-Q)}+\frac{z^2}{c^2(c^2-Q)}=0\;\ldots\ldots\;(15).$$

Also υ is a root of this equation, since, when it is substituted for Q, the equation is identical with (10), one of the equations of the line of curvature considered. Thus a line of curvature is the locus of points on the surface, for which one root of (15) is constant. Now, by combining equations (13) and (14), we have

$$\left.\begin{aligned}&\frac{a^2(b^2-c^2)(a^2-\upsilon)(a^2-p\rho)}{x^2}\\ &=\frac{b^2(c^2-a^2)(b^2-\upsilon)(b^2-p\rho)}{y^2}\\ &=\frac{c^2(a^2-b^2)(c^2-\upsilon)(c^2-p\rho)}{z^2}\end{aligned}\right\}\;\ldots\ldots\ldots\ldots\;(16).$$

These equations show that υ and $p\rho$ cannot both be constant, and therefore they must be different roots of the equation (15). Hence, if ρ' be the radius of curvature of a normal section perpendicular to the line of curvature at P, we have

$$\upsilon=p\rho'\,;$$

which shows that the radii of curvature of sections perpendicular to a line of curvature at different points, are inversely proportional to the perpendiculars from the centre upon the tangent planes at those points.

The equations (16) may be verified directly, since, $p\rho$ and υ being the two roots of (15), we have

$$\upsilon\,.\,p\rho=a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right),$$

$$\upsilon+p\rho=\frac{(b^2+c^2)\,x^2}{a^2}+\frac{(c^2+a^2)\,y^2}{b^2}+\frac{(a^2+b^2)\,z^2}{c^2}.$$

Hence $(a^2-\upsilon)(a^2-p\rho)$

$$=a^4-\{a^2(b^2+c^2)-b^2c^2\}\frac{x^2}{a^2}-\{a^2(c^2+a^2)-c^2a^2\}\frac{y^2}{b^2}-\{a^2(a^2+b^2)-a^2b^2\}\frac{z^2}{c^2}$$

$$=a^4\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)-\{a^2(b^2+c^2)-b^2c^2\}\frac{x^2}{a^2}$$

$$=(a^2-b^2)(a^2-c^2)\frac{x^2}{a^2}.$$

Hence $\dfrac{a^2(b^2-c^2)(a^2-\upsilon)(a^2-p\rho)}{x^2}=-(b^2-c^2)(c^2-a^2)\,)a^2-b^2);$

and therefore we infer that each member of (16) is equal to this expression, on account of the symmetry.

Let l', m', n' be the direction-cosines of the principal section corresponding to the root υ of the quadratic equation (15). We shall have, by the formulæ which correspond to (14),

$$l' = \frac{\mu x}{a^2 - \upsilon}, \quad m' = \frac{\mu y}{b^2 - \upsilon}, \quad n' = \frac{\mu z}{c^2 - \upsilon} \quad \ldots . (17).$$

Thus, by means of the same root of the quadratic equation, we have, in (13) and (17), expressed the direction-cosines of each of the two principal sections. If we put λ for each member of (13), these equations give

$$ll' + mm' + nn' = \lambda\mu \{a^2(b^2 - c^2) + b^2(c^2 - a^2) + c^2(a^2 - b^2)\}$$
$$= 0,$$

which proves that the principal directions of curvature are at right angles to one another; a theorem, of which many other different proofs have been given.

Dec. 1844.

VIII.—MATHEMATICAL NOTE.

1.—Light diverging from a point is incident on a given surface at a given point; find the direction of the reflected ray.

Let l, m, n be the direction cosines of the normal (N) at the point x, y, z,

$$\left.\begin{matrix} \alpha, \beta, \gamma \\ \alpha', \beta', \gamma' \end{matrix}\right\} \text{ be those of the } \left\{\begin{matrix} \text{incident ray } (I), \\ \text{reflected ray } (R), \end{matrix}\right.$$

ι the angle of incidence;

then, because (N), (I), (R) are in the same plane (f, g, h), we have

$$\left.\begin{aligned} fl + gm + hn &= 0, \\ f\alpha + g\beta + h\gamma &= 0, \\ f\alpha' + g\beta' + h\gamma' &= 0, \end{aligned}\right\} \ldots\ldots\ldots (1);$$

hence, λ, μ being indeterminate multipliers,

$$l = \lambda\alpha + \mu\alpha', \quad m = \lambda\beta + m\beta', \quad n = \lambda\gamma + \mu\gamma' \ldots . (2).$$

Also

$$\left.\begin{aligned} \cos\iota &= l\alpha + m\beta + n\gamma, \\ \cos\iota' &= l\alpha' + m\beta' + n\gamma', \\ \cos 2\iota &= \alpha\alpha' + \beta\beta' + \gamma\gamma', \end{aligned}\right\} \ldots\ldots\ldots (3);$$

hence, multiplying equations in (2) by l, m, n respectively, and adding, also by α, β, γ and α', β', γ', and subtracting the results, we get by (3)

$$1 = (\lambda + \mu)\cos\iota,$$
$$0 = \lambda - \mu,$$

or

$$\lambda = \mu = \frac{1}{2\cos\iota},$$

whence equations (2) give

$$a' = 2l \cos \iota - a,\ \ \beta' = 2m \cos \iota - \beta,\ \ \gamma' = 2n \cos \iota - \gamma \ \dots (4),$$

or equations to the reflected ray (2) are

$$\frac{\xi - x}{2l \cos \iota - a} = \frac{\eta - y}{2m \cos \iota - \beta} = \frac{\zeta - z}{2n \cos \iota - \gamma} \dots\dots (5).$$

Cor. 1. Let the incident ray pass through the origin; and let p be the length of the perpendicular from the origin on the tangent plane, r the distance of the point of incidence from the origin: then

$$\cos \iota = \frac{p}{r},\quad \frac{a}{x} = \frac{\beta}{y} = \frac{\gamma}{z} = \frac{1}{r};$$

hence equations (5) become

$$\frac{\xi - x}{2lp - x} = \frac{\eta - y}{2mp - y} = \frac{\zeta - z}{2np - z} \dots\dots\dots\dots (6).$$

Ex. Let the surface be an ellipsoid, whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

then $$l = \frac{px}{a^2},\quad m = a,\quad n = a,$$

and $$\frac{\xi - x}{\left(2\frac{p^2}{a^2} - 1\right)x} = \frac{\eta - y}{\left(2\frac{p^2}{b^2} - 1\right)y} = \frac{\zeta - z}{\left(2\frac{p^2}{c^2} - 1\right)z} \ \dots (7).$$

Cor. 2. When the reflected ray passes through the line,

$$\frac{\xi}{l_1} = \frac{\eta}{m_1} = \frac{\zeta}{n_1} = \rho \text{ suppose } \dots\dots\dots\dots (7),$$

we have, substituting in (6), and calling each fraction q,

$$\left.\begin{aligned} l_1\rho - (2lp - x)\,q &= x, \\ m_1\rho - (2mp - y)\,q &= y, \\ n_1\rho - (2np - z)\,q &= z, \end{aligned}\right\} \dots\dots\dots\dots (8);$$

and eliminating ρ and q, and substituting for l, m, n from the equation to the surface, we shall find the locus on which the rays (7) were before reflection.

Let λ_1 μ_1, ν_1 be multipliers; then putting

$$\left.\begin{aligned} \lambda_1 l_1 + \mu_1 m_1 + \nu_1 n_1 &= 0, \\ \lambda_1(2lp - x) + \mu_1(2mp - y) + \nu_1(2np - z) &= 0, \end{aligned}\right\},$$

we get $$\frac{\lambda_1}{2(m_1 n - mn_1)p - (m_1 z - n_1 y)} = \frac{\mu_1}{(\quad)} = \frac{\nu_1}{(\quad)};$$

hence, by (8),

$$(m_1 n - mn_1)x + (n_1 l - nl_1)y + (l_1 m - ml_1)z = 0;$$

or if $F(x, y, z) = 0$, be the equation to the surface, and

$$u = \frac{dF}{dx}, \quad v = \frac{dF}{dy}, \quad w = \frac{dF}{dz}, \quad \text{since } \frac{l}{u} = \frac{m}{v} = \frac{n}{w};$$

therefore the equation to the required locus is

$$(m_1 w - n_1 v)\, x + (n_1 u - l_1 w)\, y + (l_1 v - m_1 u)\, z = 0 \; \ldots \; (9).$$

Ex. In the ellipsoid this becomes

$$\left(\frac{m_1 z}{c^2} - \frac{n_1 y}{b^2}\right) x + \left(\frac{n_1 x}{a^2} - \frac{l_1 z}{c^2}\right) y + \left(\frac{l_1 y}{b^2} - \frac{m_1 x}{a^2}\right) z = 0,$$

or $$l_1 yz\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + m_1 zx\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + n_1 xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = 0,$$

the equation to a cone.

If in particular $l_1 = m_1 = n_1$, this reduces to

$$yz\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + zx\left(\frac{1}{c^2} - \frac{1}{a^2}\right) + xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right).$$

Addition.

For *refraction*, when ι' is the angle of refraction, and $\sin \iota = \kappa \sin \iota'$, we get equations (1), (2) unaltered; instead of (3) we have

$$\cos \iota = l\alpha + m\beta + n\gamma, \quad \cos \iota' = l\alpha' + m\beta' + n\gamma',$$
$$\cos (\iota - \iota') = \alpha\alpha' + \beta\beta' + \gamma\gamma';$$

whence it easily follows that

$$-\frac{\lambda}{\sin \iota'} = \frac{\mu}{\sin \iota} = \frac{1}{\sin (\iota - \iota')},$$

or $$l \sin (\iota - \iota') = -\alpha \sin \iota' + \alpha' \sin \iota;$$

and hence equations to the refracted ray

$$\frac{\xi - x}{l \sin (\iota - \iota') + \alpha \sin \iota'} = \frac{\eta - y}{m \sin (\iota - \iota') + \beta \sin \iota'}$$
$$= \frac{\zeta - z}{n \sin (\iota - \iota') + \gamma \sin \iota'} \; \ldots . \; (5').$$

These expressions give those for reflection (5) by putting $\kappa = -\iota$; $\iota = -\iota'$, as it ought to be.

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END OF VOL. IV.

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